Investigation 3.1

TANGRAM ACTIVITIES

"What's the area of this?" is probably a familiar question, particularly if *this* is a familiar figure and you know a formula for its area. But what *is* area? How can you compare two areas without using formulas or numbers? How are area and perimeter related? These are just some of the questions you will be answering in this module.

Good problem solvers need to be skilled at communicating their methods to others—in pictures, in writing, and through presentations. In addition to learning about area in this module, you will also be developing two essential mathematical habits of mind: describing a process in mathematical language, and writing an explanation that is both logically sound and clear enough to be convincing.

FOR DISCUSSION

Review what you already know about area.

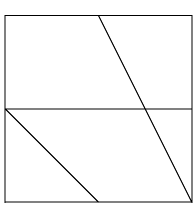
– The word *area* has a mathematical meaning as well as various everyday meanings. What *are* various meanings of *area*?

  a. Explain what *area* means in everyday language.

  b. Try to define *area* in mathematical terms.

– List some ways that you use the idea of area, both in mathematics and outside math class.

***Dissect* means the same thing as "cut up." Here is an example of a square dissected into five pieces.**



A *tangram* is a popular geometric puzzle in which a square is dissected into seven specific pieces. Tangrams and other geometric dissection puzzles are in some ways like jigsaw puzzles: the goal is to fit pieces together to make something. But they differ in two important respects. Jigsaw puzzles generally have many pieces that fit together in only one way; geometric dissections generally contain just a few pieces and fit together in more than one way.

In the next few problems, you'll use a tangram to build pictures and abstract shapes and to explore ideas about area.

Your work in this module will help you become an expert in geometric dissections.

In each problem that asks you to create a tangram shape, draw your solution or record it in some other clear way.

WORKING WITH TANGRAMS

1. Get a set of tangram puzzle pictures from your teacher. Use *all seven* tangram pieces to produce each picture.

Try more letters or numbers if you like. An attractively pasted up Tangram Alphabet Book might make a nice present for a young child.

2. Show how to arrange all seven tangram pieces into the shape of each letter in your name.

3. Use all seven tangram pieces to make a square.

4. With all the pieces, make a nonsquare rectangle.

5. Does the square you made in Problem 3 have more area, less area, or the same area as the rectangle you made in Problem 4? Explain your answer.

6. Use the two small triangles to make three shapes, each congruent to one of the other tangram pieces.

7. **a.** Compare the area of one small triangle to the area of each of the three figures you made in Problem 6.

b. Compare the areas of these three figures to each other.

8. Use the two small triangles and the medium-size triangle to make the following shapes:

a. A square;

b. A rectangle that is not a square;

c. A trapezoid;

d. A parallelogram that is not a rectangle;

e. A triangle.

9. Compare the areas of the figures in Problem 8.

10. Compare the area of the square you made in Problem 8 with the area of each of the following.

a. A small triangle in the tangram set

b. The medium-size triangle in the tangram set

c. The square in the tangram set

11. **a.** Compare the areas of the medium and large triangles.

b. Compare the areas of the small and large triangles.

FOR DISCUSSION

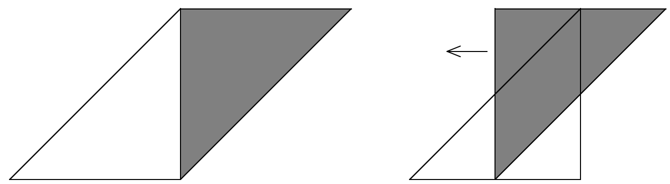
What ways of thinking about area did you use in these problems? List and discuss some different ways to compare the areas of two different shapes.

MOVING THE PIECES

One set of pieces can make more than one shape. Geometric language helps you give clear descriptions of how you move pieces to change one shape to another. You will use three basic moves:

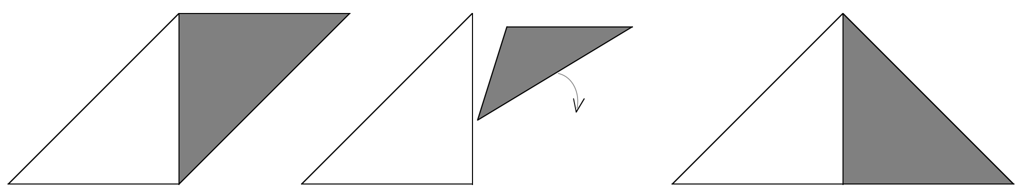
SLIDE OR TRANSLATION

Here, a parallelogram becomes a square as one of the small triangles slides parallel to the base of the parallelogram. Try this with your tangram pieces.



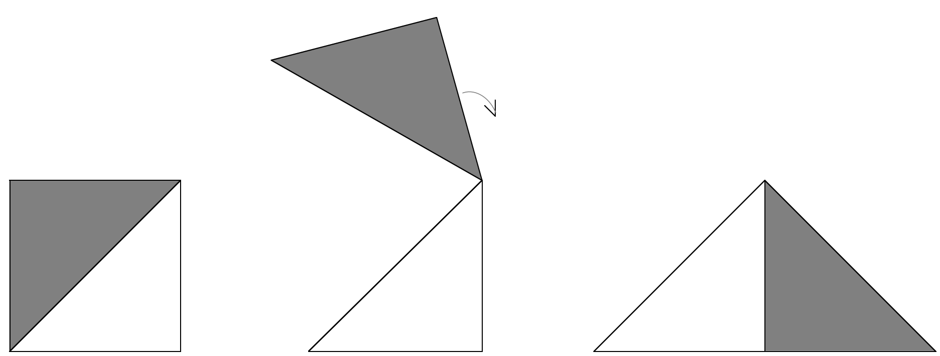
FLIP OR REFLECTION

Here, the parallelogram becomes a triangle when one of its halves (a triangle) is flipped. Try it.



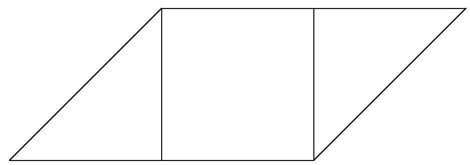
ROTATION

A piece can be rotated around a point. Here, a square becomes a triangle when one of the small triangles is rotated around a vertex. Try it.



12. For each of the following pairs of figures:

- build the shape on the left with your tangram set;
- turn it into the shape on the right by reflecting, rotating, or translating one or two of the pieces (this may take more than one step);
- write a description, telling which piece or pieces you moved and how you moved them.



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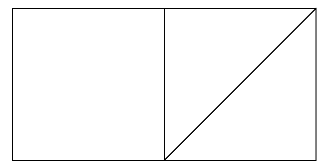

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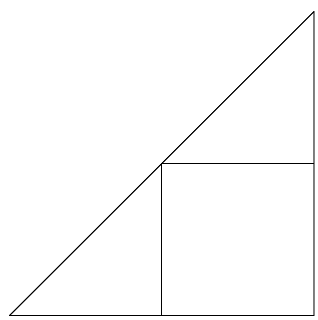

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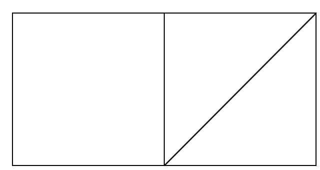

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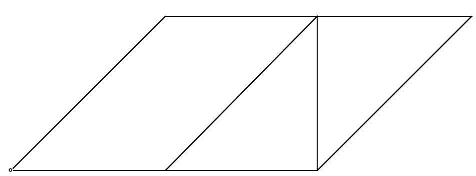

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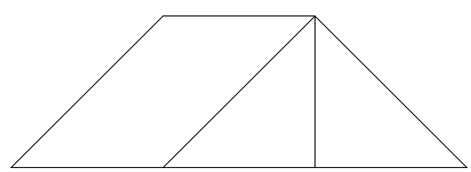
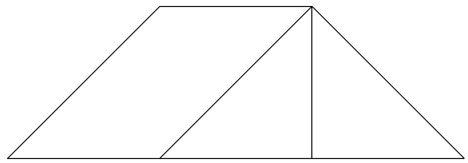


$\mathbf{c_2}$

CHECKPOINT

13. Sabrina made the following shapes with her tangram set. Which shapes have the same area? How do you know?



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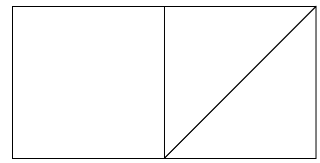

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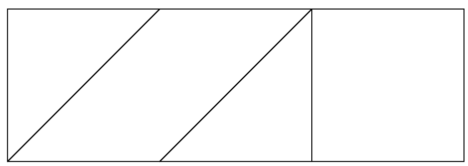

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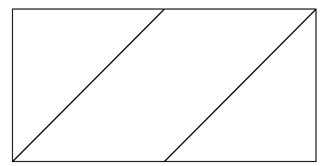

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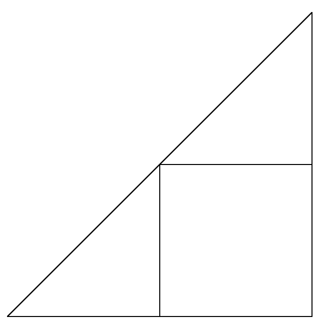

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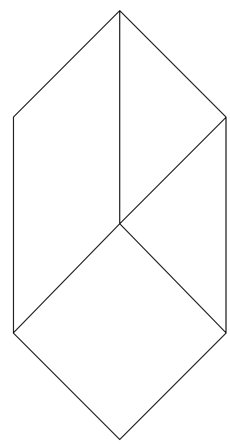

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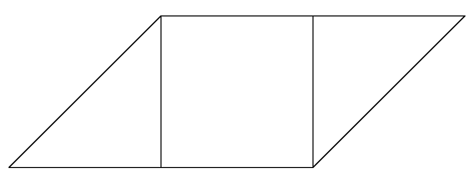

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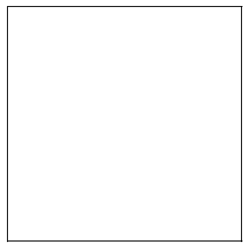
14. Which of the following figures do you suspect have the same area? Explain your answer.



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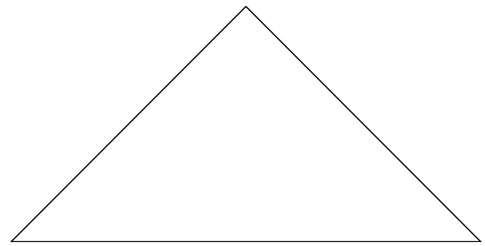

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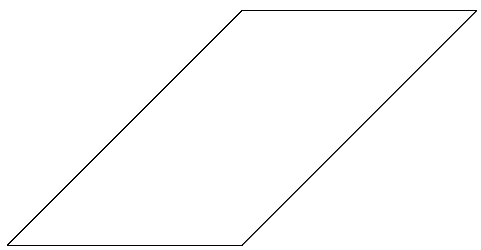

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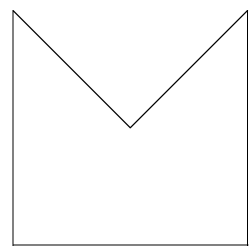

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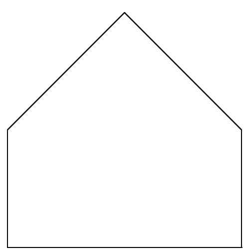

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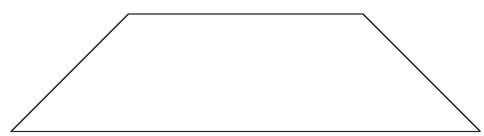

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15. Write and Reflect Decide whether each of the following statements is true or false and give your reasons. (If it is false, a single counterexample is good enough reason.)

- If two figures are congruent, they have the same area.
- If two figures have the same area, they are congruent.

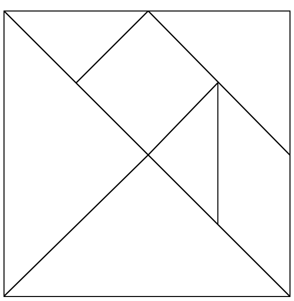
TAKE IT FURTHER

16. **a.** You have already made a square from a single tangram set. Can you make a square using all 14 pieces from two tangram sets?

b. For which numbers *n* can you make a square using all the pieces from *n* sets of tangrams? For which numbers is it *not* possible?

PERSPECTIVE ON THE HISTORY OF THE TANGRAM

Tangrams have a long and interesting history. This essay will introduce you to their story.



If you have used tangrams before, you may have heard some version of this story:

> The tangram is an ancient puzzle from China that has been known there for thousands of years. The puzzle originated when old Mr. Tan (or Tang) was walking home, carrying a beautiful square tile. He stumbled on a rock and dropped the tile, which broke into the seven geometric shapes you find in your puzzle. While trying to fit the pieces back into the original square, he became fascinated with the many other shapes he could create.

Or perhaps you heard the story popularized by Sam Loyd:

> Tangrams originated in China 4000 years ago in a seven-volume set of books entitled *The Seven Books of Tan*. One book, found in Peking (now Beijing), was printed in gold leaf on parchment.

Alas, these stories are just legend. Karen Dee Michalowicz, a middle school mathematics teacher in Virginia, has researched many historical topics in mathematics. According to her research, tangrams were popular puzzles in the 19th and 20th centuries in Europe and America, but there is no evidence that the puzzles existed before the early 1800s. The story about Mr. Tan is just that—a story, intended to entertain.

Sam Loyd published a puzzle book in 1903 called *The 8th Book of Tan*, which included the tangram puzzle. To increase sales, or maybe for his own amusement, he made up the story of *The Seven Books of Tan*. In the year 2000 B.C., the Chinese could carve pictures into stones and pottery, but they had no parchment and no facilities to print in gold leaf. Amazingly, this story was widely believed and passed on; so much so that teachers today still share it with their students as if it were true.

Possibly because these fantastic accounts are so appealing, they have survived and been passed on, while little is known about the true origins of the puzzle. Tangrams may have originated in China. If so, the puzzle may have been brought back to Europe by travelers to China, but surely no more than 200 years ago. The Chinese name for tangrams is “ch’i ch’iao t’u,” which, translated literally, means “seven-ingenious plan.”

One well-recorded piece of tangram history is their appeal. Of course, puzzlers like Sam Loyd and H. E. Dudeney loved their tangrams. It is rumored that Napoleon had a tangram set and that he spent a lot of time playing with it after his exile. It would not be surprising if the rumor were true: Napoleon’s obsession with mathematics, and geometry in particular, is well-known. The ivory tangram set that was a favorite of poet and writer Edgar Allen Poe is now owned by the New York Public Library. Lewis Carroll often designed his own tangram puzzles, and he owned one of the earliest books of tangram puzzles. Both Poe and Carroll, like Karen Michalowicz, were known to love mathematics and games of logic.

Investigation 3.2

CUT AND REARRANGE

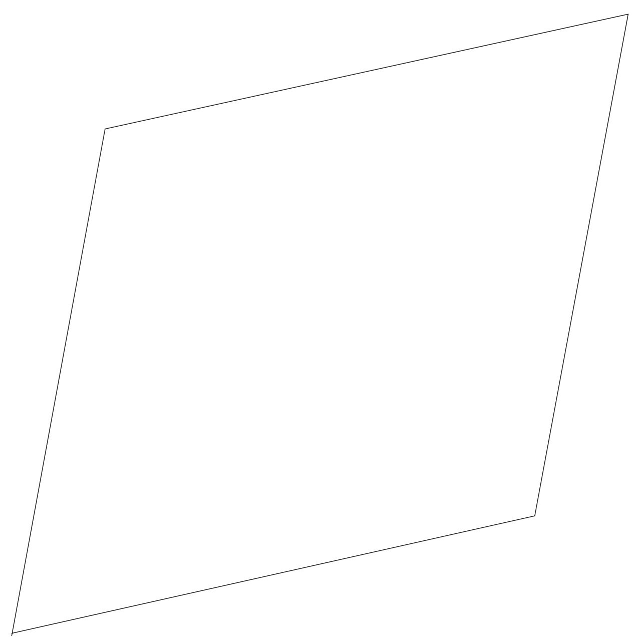
In this investigation, you will change one shape into another by cutting the first shape into parts and rearranging the parts to form the second shape. Trace and cut out a copy of the first shape. Then, cut that copy into pieces and rearrange the pieces (be sure to use all of them!) to make a new shape. Because the two shapes will be made from the same pieces, they will be equal in area.

Cutting and rearranging will not convert a given shape into just *any* other imaginable shape. For example, you would not expect it to work if the resulting shape had more or less total "stuff" in it! Chih-Han Sah, a mathematician who specializes in dissections of this sort, calls two figures that *can* be cut into each other "scissors-congruent."

CUTTING PROBLEMS

Each of the following problems can be solved in more than one way. Work with others to find a few solutions. Then, pick one that you like best. *Save your written work. You will need it later.*

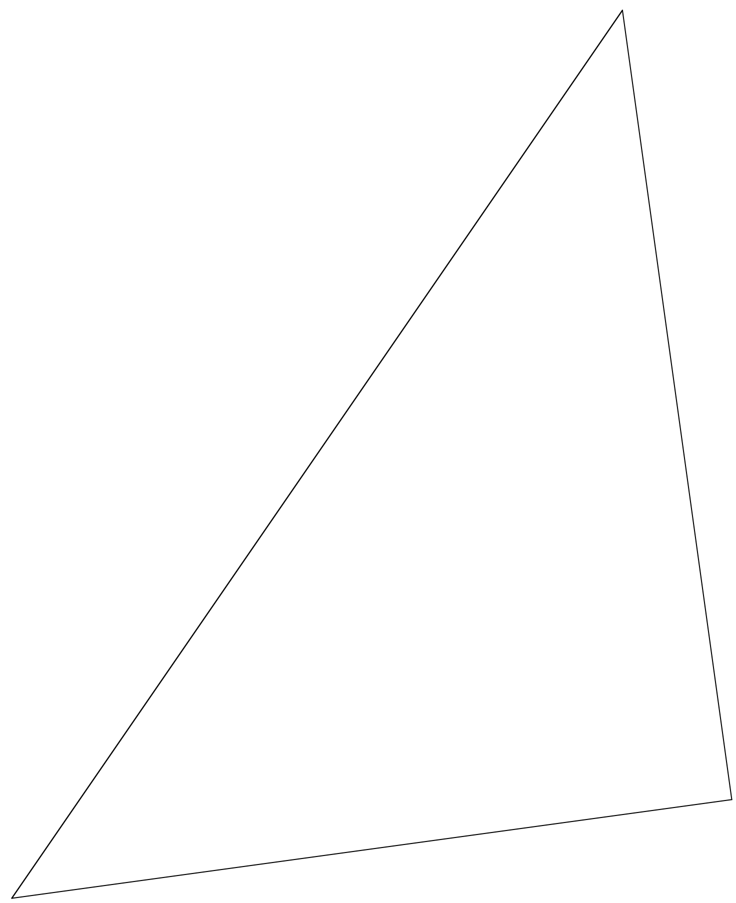
1. a. Trace this parallelogram. Find a way to cut your copy into pieces that can be rearranged to form a "scissors-congruent" rectangle.



b. Write a description of the cuts you made and of how you moved the pieces.

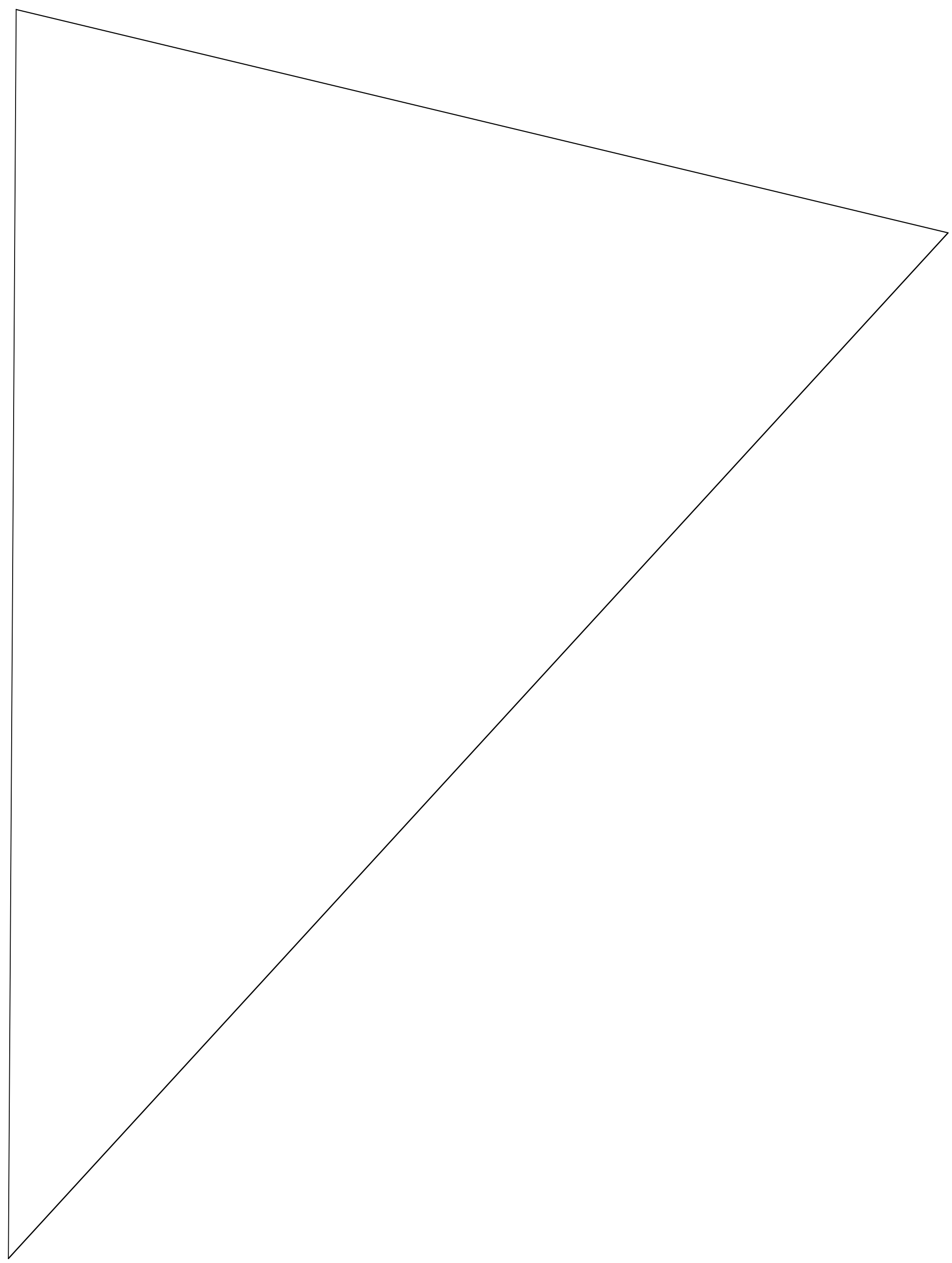
2. a. Copy this right triangle, then dissect the copy so that the pieces can be rearranged to form a rectangle.

b. Write a description of the cuts you made and of how you moved the pieces.



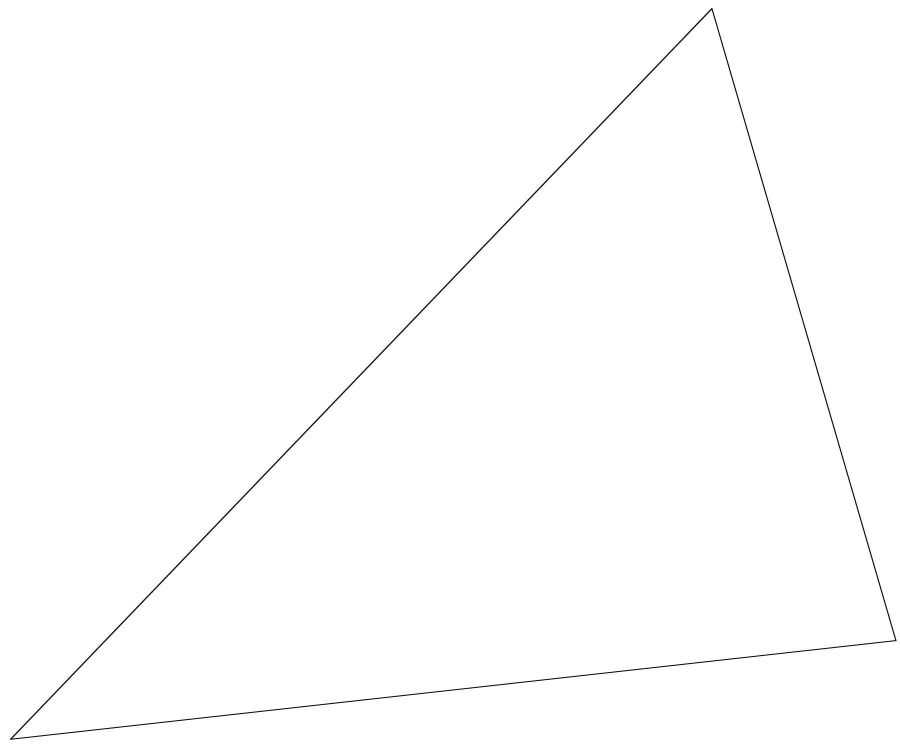
3. **a.** Dissect a copy of this triangle into pieces that will form a parallelogram.

b. Write a description of the cuts you made and of how you moved the pieces.



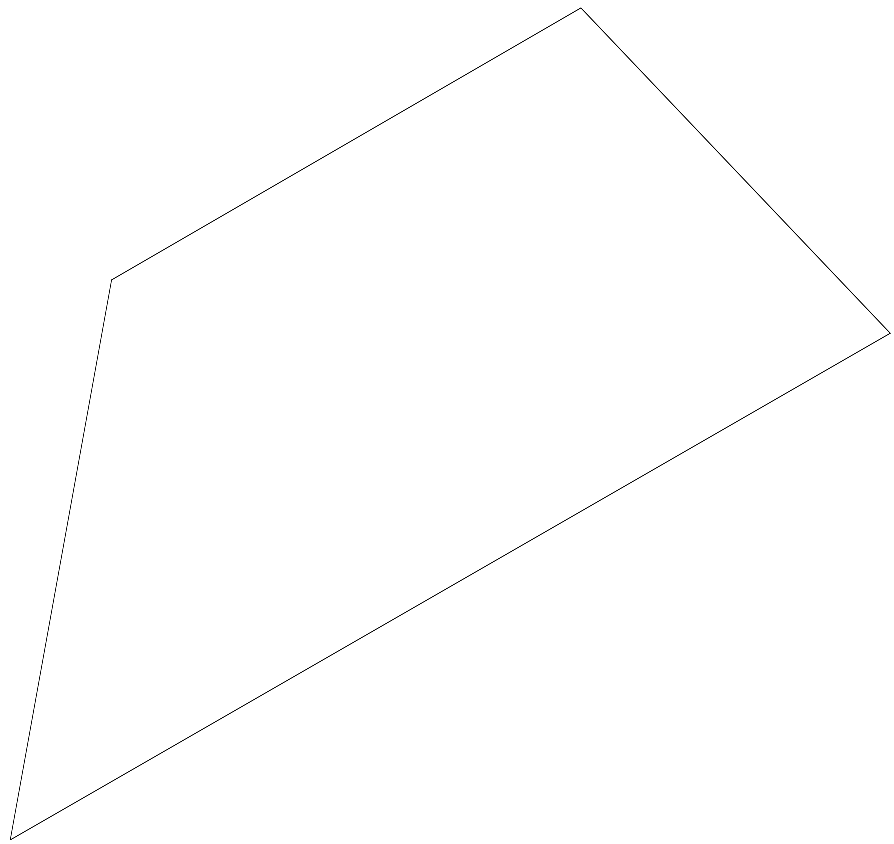
4. **a.** Dissect a copy of this triangle into pieces that will form a rectangle.

b. Write a description of the cuts you made and of how you moved the pieces.



5. **a.** Dissect a copy of this trapezoid into pieces that will form a rectangle.

b. Write a description of the cuts you made and of how you moved the pieces.



6. **a.** Make a second copy of the trapezoid above, and dissect it so that the pieces form a triangle.

b. Write a description of the cuts you made and of how you moved the pieces.

7. Can you reverse a dissection process? Choose three shapes from the list below. For each of the shapes you chose, trace the rectangle on page 15 and dissect it into pieces you can rearrange to form that shape.

a. An isosceles triangle

b. A right triangle

c. A nonrectangular parallelogram

d. A scalene triangle

e. A trapezoid

DO THE CUTS REALLY WORK?

Have you ever expected a dissection to work, but then discovered that the pieces didn't quite fit? Or perhaps the pieces *looked* like they fit, but it was difficult to be sure? (See Investigation 3.13 for an example.) When you can explain *why* a cut works, you can know for sure that it does.

PROPERTIES OF PARALLELOGRAMS

When mathematicians talk about *dissecting*, they often don't stop at cutting up a shape—they want to rearrange the pieces into something new!

To understand why a dissection works, you must know *properties* of the shapes you're cutting. Start with parallelograms.

8. Write and Reflect List all of the properties of parallelograms that you can think of. The list is started below; see how many properties you can add to it.

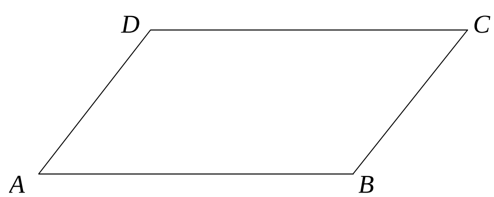
- Parallelograms have exactly four sides.
- Opposite sides are parallel.

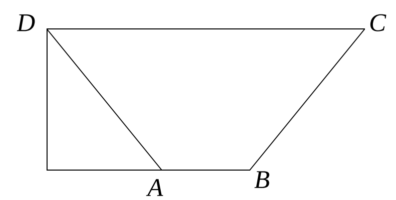
Share your ideas with others. You'll need as complete a list as possible to help you answer the next few questions.

PARALLELOGRAMS TO RECTANGLES

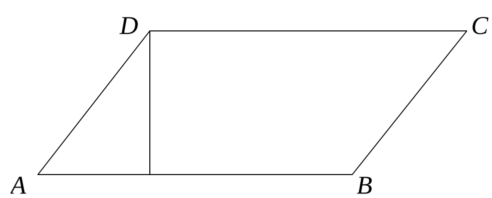
Here is one student's method for dissecting a parallelogram into a rectangle:

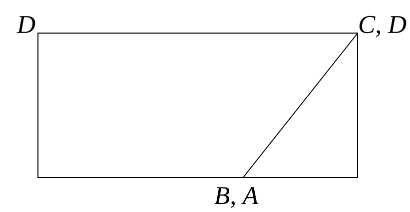
> First, cut out the parallelogram. Then fold over vertex A, so you fold at vertex D and A lines up on $\overline{AB}$.



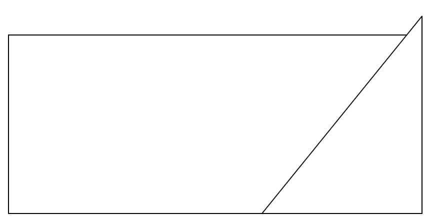


> Then unfold, cut along the crease, and slide the triangular piece along $\overline{AB}$ so that it matches up with the other side of the parallelogram. $\overline{AD}$ matches up with $\overline{BC}$. Then you have a rectangle.

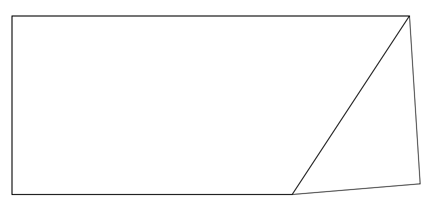




The student's cut created two pieces: a triangle and a trapezoid. The student then described how to rearrange these pieces. But what guarantees that the rearrangement has four sides? Here are two ways that the dissection might fail.



The newly "glued" edges might not match.



The newly created bottom might be bent.

9. Explain how the properties of parallelograms assure that when you slide the triangle to the opposite side of the trapezoid,

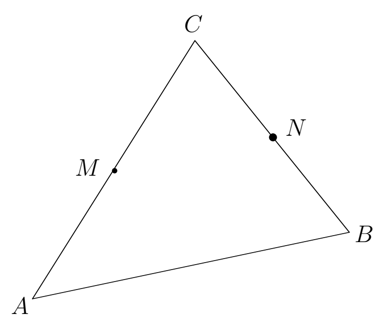
a. the two pieces will fit together exactly;

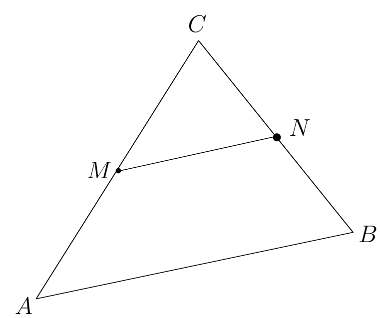
b. the new bottom edge will be straight.

TRIANGLES TO PARALLELOGRAMS

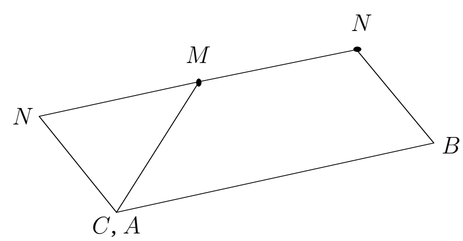
Here is a student's method for dissecting a triangle into a parallelogram.

First, cut out the triangle and find the midpoints of $\overline{AC}$ and $\overline{BC}$. Let M and N be the midpoints of $\overline{AC}$ and $\overline{BC}$, respectively. Cut along $\overline{MN}$.



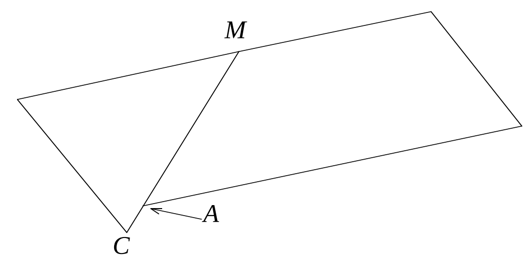


Rotate $\triangle MCN$ around M until $\overline{MC}$ matches up with $\overline{MA}$. Then we have a parallelogram.

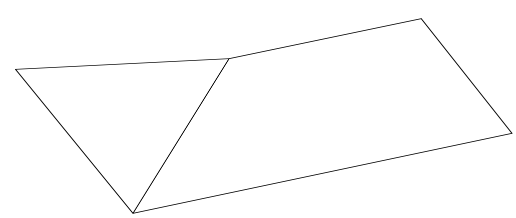


Use the given pictures and descriptions, along with your list of properties of parallelograms to answer the questions below.

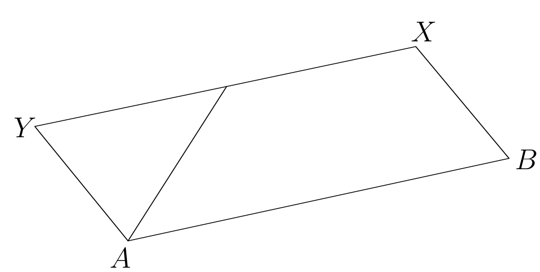
10. If the final figure is to have just four sides, then segments $\overline{MC}$ and $\overline{MA}$ must be congruent and not look like the picture below. What part of the student's procedure guarantees that these segments match?



11. If the final figure is really a parallelogram, then the top must be straight and not look like the picture below. How does the student's method *guarantee* that it will be straight?



12. If the final figure is really a parallelogram, then in the picture below, it must be true that $\overline{AY} \cong \overline{BX}$. How does the student's method *guarantee* that $\overline{AY} \cong \overline{BX}$?



PROPERTIES OF RECTANGLES

You probably know even more about the properties of rectangles than you do about the properties of parallelograms. Rectangles are even more special than parallelograms—they are a special type of parallelogram—and so they will have more properties on their list.

After saying that rectangles are parallelograms, isn't it redundant to repeat that they, too, have four sides? Yes. But that's not so bad. Listing *all* the properties—even "obvious" or redundant ones—can help you notice things you might otherwise overlook.

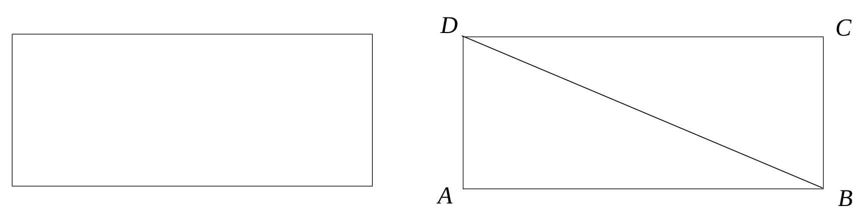
13. Write and Reflect Write a list of all the properties of rectangles that you can think of. The list is started for you below. See how many properties you can add.

- Rectangles are parallelograms.
- Rectangles have exactly four sides.
- All angles measure 90°.

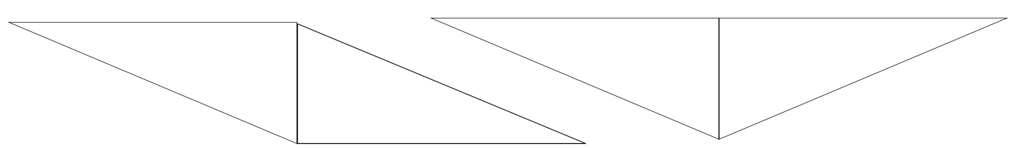
RECTANGLES TO TRIANGLES

Here is one student's method for turning a rectangle into an isosceles triangle:

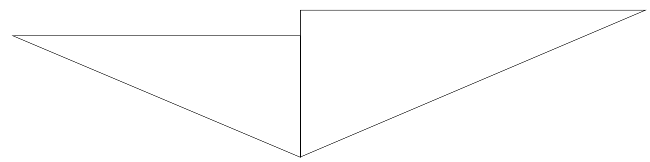
Start with a rectangle. Cut along a diagonal.



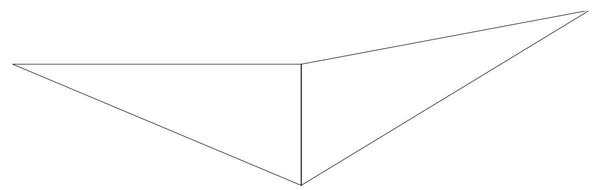
Slide $\triangle ABD$ to the right, along the bottom, so $\overline{AD}$ lines up with $\overline{BC}$. Then flip $\triangle ABD$ so you have a triangle instead of a parallelogram.



The student's diagonal cut creates two triangles. What guarantees that the final rearrangement of these pieces has three sides? Here are two ways that the dissection might fail.



The newly "glued" edges might not match.



The newly created top might be bent.

14. Explain how the properties of rectangles assure you that when you slide $\triangle ABD$ to the right and flip it according to the student's instructions,

a. $\overline{AD}$ will fit $\overline{BC}$ exactly;

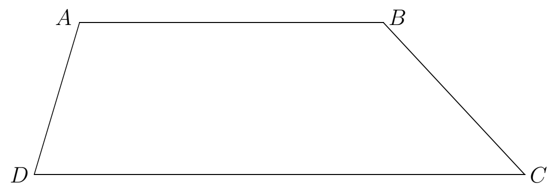
b. the new top edge will be straight.

15. If you cut along the other diagonal, will you get a different triangle? Explain your answer, being sure to use what you know about rectangles.

CHECKPOINT

16. The following is a copy of the answer that one group of students wrote for Problem 5. From this explanation, figure out what the students must have done. Then rewrite the explanation to clarify it. Add mathematical language where it is useful.

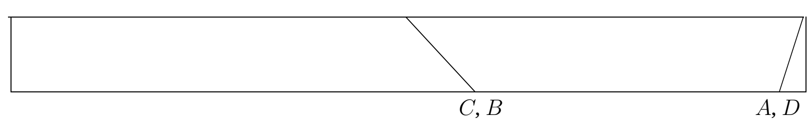
First, we started out with a trapezoid.



Next, we folded $\overline{AB}$ down to $\overline{CD}$.

Then we cut on the line that we folded.

Then, we folded the left end corner and cut it to make the other side a straight end, so it would look like a rectangle.

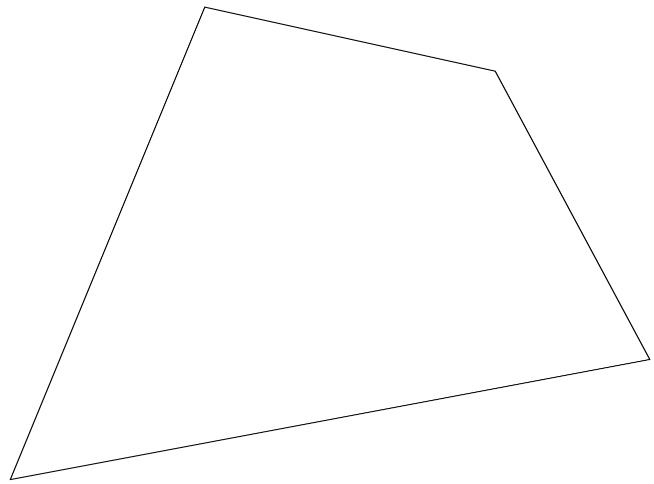


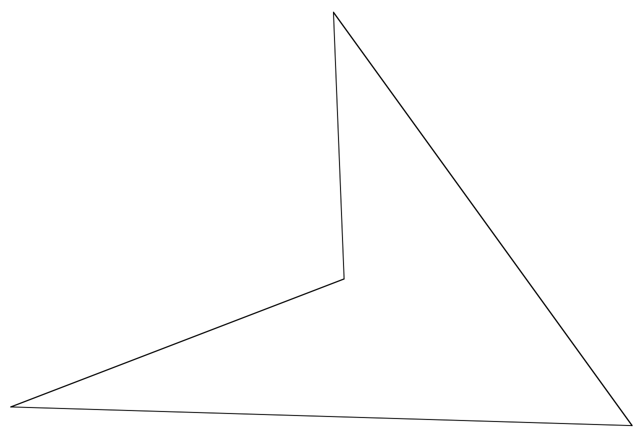
17. You have been showing that two different shapes can have equal area by finding a way to cut up one and rearrange its pieces to form the other. If you suspected that two particular shapes had *unequal* area, but you did not know area formulas, how could you show that one had a greater area than the other?

TAKE IT FURTHER. . . . .

Can an arbitrary quadrilateral be dissected into a rectangle?

18. Approach the problem experimentally. Trace the two figures below, and then try to cut them into rectangles.





If your experiment succeeded, you have shown that you can cut *these two* particular quadrilaterals into rectangles. And, if the experiment failed, could there not be *some* method—perhaps a very complicated one—that might succeed?

19. Try to come up with a *reasoned argument* to say why any quadrilateral can or cannot be dissected into rectangles.

Investigation 3.3

Cutting Algorithms

An *algorithm* is a process—a set of steps—that is completely predetermined. There is no unpredictability, no dice roll, no doubt that it will do precisely what it did the previous time. There are also no hard judgment calls—no "more or less" about it—just clear, explicit instructions. Many computer programs are written as algorithms, with every step precisely spelled out.

A process can be quite reliable without doing what you want done. Being reliable and predictable is not the same as being successful. A successful algorithm can be used to solve a problem, to make something, or to accomplish a task. For example, you probably know an algorithm for adding two fractions with different denominators. There is often more than one algorithm for a particular purpose.

The word *algorithm* is a distorted transliteration of a great Islamic mathematician's name. abu-Ja'far Muhammad ibn Musa was known as al-Khwarizmi, which means "the one from Khwarazm." The Latin attempt at spelling al-Khwarizmi was Algorismi, which later became *algorism* and then *algorithm*.

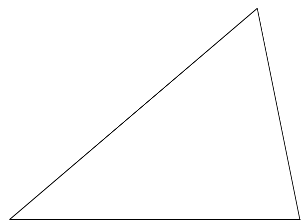
In Problems 1 through 6 from Investigation 3.2, you were asked to "Write a description of the cuts you made and of how you moved the pieces." You may have written your descriptions as algorithms.

Now you will develop *general* algorithms for successfully dissecting *any* triangle, trapezoid, or parallelogram into a rectangle and you will prove that your algorithms will always work. Remember that, for each problem, there may be very many algorithms that work, but there are many more that don't or that work only for special cases.

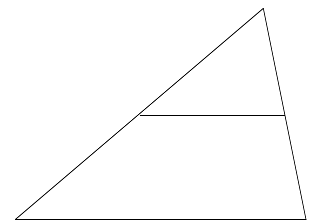
Later, you will use successful algorithms to develop formulas for the area of triangles, parallelograms, and trapezoids.

DESCRIBING THE STEPS

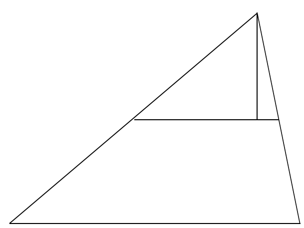
1. The pictures below show a triangle being dissected into a rectangle. Write complete and precise instructions for the cuts and rearrangements that are illustrated. Try to make each step clear enough for someone else to follow.



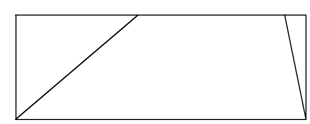
a



b



c



d

You've done a lot of thinking since Problems 1, 4, and 5 in Investigation 3.2, where you first wrote about how you dissected a parallelogram, triangle, and trapezoid into rectangles. The next problems ask you to edit these old descriptions (or write new ones) to be as clear and precise about the steps as you now can be.

2. For each dissection—parallelogram, triangle, and trapezoid to rectangle—use two pages to rewrite your algorithm.

a. On one page, draw pictures that illustrate the steps of your method.

b. *On the other page*, describe the same steps precisely using *words only*.

3. Find a partner. Exchange just the words you have written (*no pictures*) describing how to dissect a parallelogram into a scissors-congruent rectangle.

- **a.** Follow the directions you get *exactly as they are written* to try out your partner's algorithm. Does the algorithm work? Are the directions clear? Were any directions confusing?
- **b.** Give your partner feedback on the algorithm you tried, and listen to your partner's feedback on yours. How can the instructions be refined?

4. Work with a partner in the same way to refine your algorithms for dissecting a trapezoid to a rectangle.

5. **Write and Reflect** Based on your experiences giving and getting feedback on two of the algorithms, rewrite all three of your dissection algorithms to be as clear as possible. Write final versions of your algorithms (with pictures if you like) for

- **a.** dissecting a triangle into a rectangle;
- **b.** dissecting a parallelogram into a rectangle;
- **c.** dissecting a trapezoid into a rectangle.

As before, carefully save this work.

CHECKING AN ALGORITHM

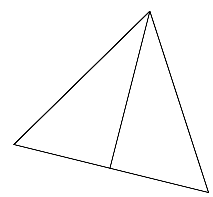
Sometimes the best way to know whether an algorithm works for all possible cases is to try it on all possible cases. That won't work for the three algorithms you just wrote. (Why not?) In cases like these, you need another way to check. Sometimes the best approach is to check the justification for each step. When that seems too difficult, you may begin by looking for counterexamples, that is, special cases in which the algorithm will fail. But how do you know where to look for special cases when there are an infinite number of cases from which to choose?

You may get some ideas from the next problem, which gives an algorithm for dissecting a triangle into a rectangle. This algorithm works perfectly for some triangles (in fact, for an infinite number of triangles!) but not for *all* triangles.

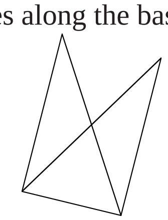
6. Study the pictures and justifications for each step of the following algorithm.

ALGORITHM

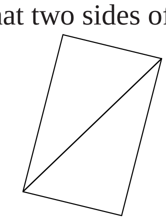
First cut out the triangle and draw an altitude from the top vertex to the base. This makes two right triangles because the altitude forms 90° angles.



Then slide one of the triangles along the base.



Flip one of the triangles so that two sides of the original triangle match up.



The final shape is a rectangle because

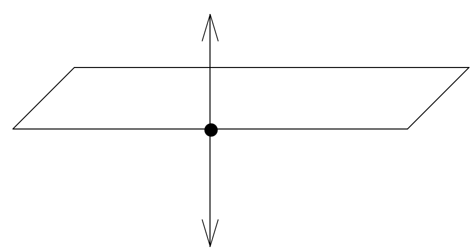
- the two vertical sides are congruent since they were made by the same cut, and
- two opposite angles are right angles since they were made by an altitude cut.

a. Show why this method won't work for all triangles. (For example, you can give an example of a triangle for which it won't work, and explain how it will fail.)

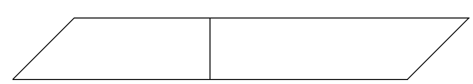
b. Explain what special feature or features a triangle must have to allow this method to work.

7. Here is another algorithm, this time for cutting a parallelogram into a rectangle. This also has a problem, but it is quite subtle. It seems to work perfectly, but there are parallelograms for which it fails. Fix it.

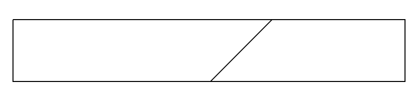
Draw the perpendicular bisector of one side of the parallelogram.



Cut along the perpendicular bisector. This perpendicular cut guarantees right angles.



Slide one piece parallel to the bisected side until the uncut ends match.



The sides *will* match—the properties of parallelograms guarantee that opposite sides are congruent and that the sum of measures of consecutive angles is 180°.

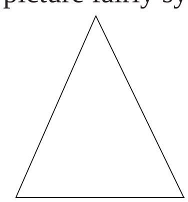
FOR DISCUSSION

This investigation began by asking "how do you know where to look for special cases when there's an infinite number of cases from which to choose?"

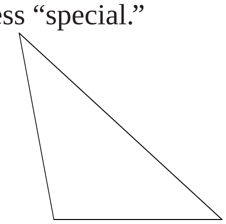
Think about your own thinking. How did you solve the last two problems? What ideas did you try? Did you start by thinking about the steps of the algorithm, or did you start by thinking about cases that might fail, or did you have some other approach? How many different good approaches did your class generate?

One very natural pitfall in writing an algorithm is basing it on a drawing or a situation that is too "standard." Even very experienced people do it, and they do it often.

For example, most people tend to picture fairly symmetric figures.

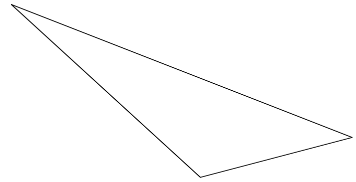


But a less symmetric figure is less "special."



Most people also tend to picture the figures with one side horizontal, so that they look like they won't "fall over." That also is too standard.

A rule that works well for a "stable" figure may not work so well for one that looks "unbalanced."



This is why it is so important to have good strategies for checking and *debugging* a process.

Have you ever wondered why computer programmers talk about "debugging" a program. Here's the answer!

The word *debug* is in common use today; it means to fix the problems in an algorithm (usually a computer program). The term is derived from the phrase "a bug in the program" or "a bug in the system," which is widely attributed to Grace Hopper, a pioneer in the computer field. She was once described as "the third programmer on the first computer in the United States." She was also the inventor of COBOL, one of the first computer programming languages.

Grace Hopper worked on huge computers. Some were the size of a room and were filled with moving parts. One of her programs was not running correctly, and she couldn't figure out why. Finally, she went into the computer and found a large moth in the machinery. When she removed this bug from the system, her program ran well, and computer scientists everywhere had a new phrase to use.

FOR DISCUSSION

One way of finding important special cases with which to test an algorithm is to stretch a definition to its "extreme."

- Discuss the difference between the *definition* of a trapezoid and your usual *mental picture* of a trapezoid.
- With other students, find three or four trapezoids that are very "nonstandard" and also quite different from each other.

8. Write and Reflect Study your algorithms for dissecting parallelograms, triangles, and trapezoids into rectangles. Check some nonstandard cases for each algorithm. If there *are* cases for which your algorithm does not work, explain what *feature* (or *features*) of them foiled your algorithm. And, of course, the mathematical thing to do is revise the method and test it again.

JUSTIFYING THE CUTS

You now have algorithms to cut and rearrange triangles, parallelograms, and trapezoids into scissors-congruent rectangles, and you are reasonably convinced that these algorithms work in all cases. It's time to justify your work geometrically.

What Does It Take to Make a Rectangle? In Investigation 3.2, you listed some properties of rectangles. To be sure that a given shape is a rectangle, you *could* check every property you know, but that's probably not necessary.

9. Look back at your list of properties of rectangles from Problem 13 in Investigation 3.2. Suppose you have a figure, and you want to know if it is a rectangle.

a. Is there any single property on your list that is enough for you to check? If so, which one?

b. Are there any pairs of properties that are enough to guarantee you have a rectangle? Which pairs?

To answer these questions, you may want to make hand drawings or use your geometry software. In constructing figures, see if the given information forces you to draw a rectangle or not. If you can make a figure that meets the criteria but is not a rectangle, then the given information is not sufficient.

10. Which properties of rectangles do you think are easiest to check from cutting and rearranging? Make a list of the properties that are easiest to check, and some sets of them that guarantee you have a rectangle.

11. **Write and Reflect** Review your algorithms once more. Justify each step to show why your sequence of steps reliably produces the desired result. Why do pieces match up? Why are line segments straight? Why are angles right angles? Once more, pay close attention to see if the justification you use is true in general, or only in certain cases.

CHECKPOINT

12. Define each of the following terms.

- **a.** algorithm
- **b.** trapezoid
- **c.** dissection
- **d.** perpendicular bisector
- **e.** debug

13. Write an algorithm for dissecting a parallelogram into a triangle.

14. Write an algorithm for dissecting a triangle that is isosceles (but not equilateral) and rearranging it into a scalene triangle.

TAKE IT FURTHER

15. Give an algorithm for dissecting a triangle into a triangle congruent to the original one.

16. In Problem 1 of this investigation, you described how to dissect a triangle into a rectangle. Suppose someone followed your algorithm and then handed you the rectangle. How would you dissect the rectangle back into the original triangle?

17. Suppose that Jane has an algorithm for dissecting a trapezoid into a rectangle. Explain how you could use Jane's steps to dissect a rectangle into a trapezoid.

18. Take someone's algorithm for dissecting a trapezoid into a rectangle, and take someone else's algorithm for dissecting a rectangle into a parallelogram. Use these two algorithms to create an algorithm for dissecting a trapezoid into a parallelogram.

19. Take someone's algorithm for dissecting a trapezoid into a rectangle, and someone else's algorithm for dissecting a triangle into a rectangle. Use these two algorithms to create an algorithm for dissecting a trapezoid into a triangle.

Investigation 3.4

AREA FORMULAS

You probably know the area formula for a rectangle. It no longer matters whether you *remember* the area formulas for triangles, parallelograms, and trapezoids. By formalizing the cutting and rearranging ideas you have developed, you are ready to discover these formulas on your own, and you can prove that they're correct.

PARALLELOGRAMS

The first problems in Investigation 3.2 asked you to dissect a parallelogram into a rectangle. By the end of Investigation 3.3, you had refined your dissection into a *general* solution that reliably rearranges *any* parallelogram, no matter what its dimensions, into a scissors-congruent rectangle. In the next few problems, you will use your dissection to discover an area formula for parallelograms.

1. Make a parallelogram, and dissect it into a rectangle using an algorithm that has been proven to work with any parallelogram.

a. Measure and record the length and width of the rectangle.

b. As you carefully rearrange your pieces to reconstruct your original parallelogram, notice what measurements on the parallelogram correspond to the rectangle's length and width.

c. Discuss your results with others. What conjectures do you make about measurements on a parallelogram compared with the dimensions of a scissors-congruent rectangle?

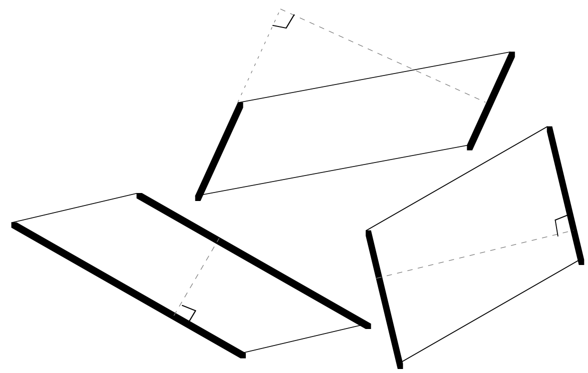
You have lots of data and no counterexamples, but your observation is a conjecture until it is proved for *all* possible parallelograms.

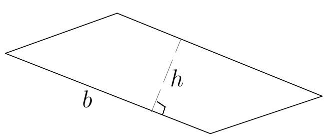
Is there a difference between the image you have of the "height of a parallelogram" and this formal definition?

DEFINITION

The ***height*** of a parallelogram is the perpendicular distance (shown as dashed lines in the following figures) between any pair of parallel sides chosen as the bases (the heavy lines).

The height is not the same as the length of the remaining sides, unless the parallelogram happens to be rectangular. Depending on the choice of the bases, the height must sometimes be measured outside or partly outside the figure.





In your dissection, does it matter which side of the parallelogram you call the base?

2. Draw a new parallelogram, labeling a base b and the corresponding height h.

a. Dissect the parallelogram and rearrange the pieces into a rectangle.

b. Explain how your dissection ensures that the base and height of the parallelogram are the same as the base and height of the rectangle. Your explanation shouldn't use any numbers.

3. Based on your work, and on what you know about the area of rectangles, find a formula for the area of a parallelogram.

TRIANGLES

In Investigation 3.3, you worked to develop an algorithm for dissecting *any* triangle into a scissors-congruent rectangle. From any correct algorithm, you can discover an area formula for triangles.

4. Draw a triangle, and dissect it into a rectangle using an algorithm that has been proven to work with any triangle.

a. Record the length and width of the rectangle.

b. Carefully rearrange your pieces to reconstruct the triangle, noticing how the rectangle's length and width relate to the triangle.

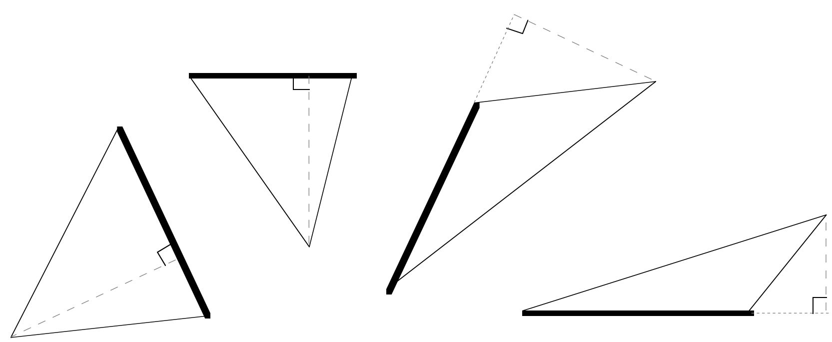
c. Compare your thinking with that of some of your classmates. Write a conjecture about how measurements on a triangle compare to the dimensions of a scissors-congruent rectangle.

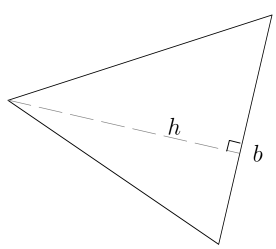
DEFINITION

Designate any side of a triangle as its base (shown as a heavy line). The ***altitude*** (height) of the triangle to that base is the perpendicular distance (shown by a dashed line) between that base (or the line containing it) and the opposite vertex.

How do you normally *picture* the "altitude of a triangle"? How is the formal definition the same as your "picture"? How is it different?

Depending on the choice of the base, the altitude sometimes must be measured outside the figure.





5. Draw a new triangle, and choose a base so that you can sketch the altitude *inside* the triangle. Label this base and the corresponding height.

a. Dissect your triangle into a rectangle.

b. Explain what happens to the base and height at each stage of the dissection. What are the dimensions of the rectangle in terms of b and h?

6. Based on your work, find a formula for the area of a triangle.

Does it matter which side of the triangle you choose as base?

TRAPEZOIDS

In Investigation 3.2, you dissected a trapezoid into a rectangle. In designing an area formula for trapezoids, you may want to use that algorithm and what you know about rectangles. Or, you might choose to dissect a trapezoid into a parallelogram or triangle and use what you now know about them.

FOR DISCUSSION

Decide on a method for turning a trapezoid into some other figure for which you know an area formula. Consider the ease of dissection and of comparing the dimensions.

7. Draw a trapezoid (you might try one with strange dimensions), and trace at least two copies of it to dissect.

a. Dissect the trapezoid into another figure whose area you know how to compute.

b. Measure and record the dimensions of the new figure that you need to calculate its area.

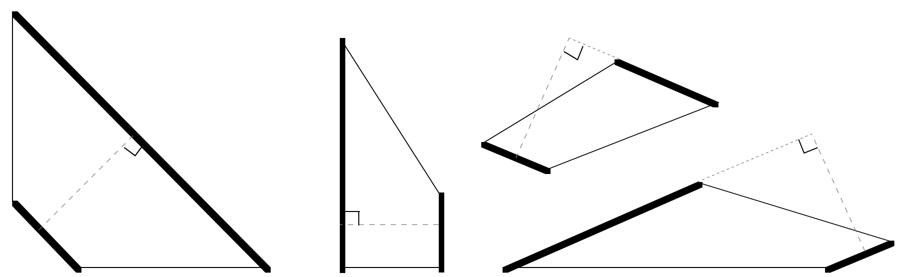
c. Carefully rearrange your pieces to reconstruct the trapezoid, noticing how the dimensions of the new figure compare to the dimensions of the trapezoid.

d. How do measurements on the trapezoid compare to the dimensions of the new figure you made?

How do you normally *picture* the "base of a trapezoid"? How is the formal definition the same as your "picture"? How is it different?

DEFINITION

The two parallel sides (shown as heavy lines) of a trapezoid are called its ***bases***. The ***altitude*** (height) of the trapezoid is the perpendicular distance (shown by the dashed line(s)) between the bases.



8. Now make it formal. Using methods similar to those you used for parallelograms and triangles, find a formula for the area of a trapezoid.

CIRCLES

FOR DISCUSSION

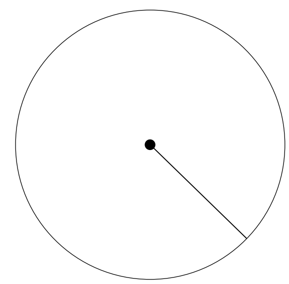
Can a circle be dissected (with a finite number of cuts) into a rectangle?

You can get useful *ideas* about the area of a circle by thinking about how it might be dissected into a rectangle, even if it would take more cuts than you could ever actually perform.

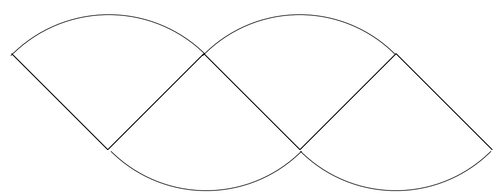
Below is a set of dissections that suggests a way to think about the area of a circle. Read it carefully and go through the process yourself, performing each dissection. As you work, try to figure out for yourself where this process is going, and what line of thinking might lead you to develop an area formula.

The problems at the very end of this section will lead you toward that line of thinking; if you like thinking on your own, don't look too far ahead.

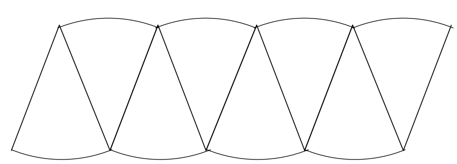
You will need at least three copies of a circle to dissect in this experiment. You can copy this one, or make your own with a compass or computer.



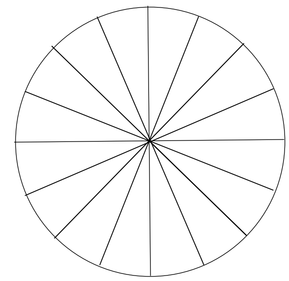
Step 1: Cut one circle into four congruent wedges, and arrange them as shown. Tape the pieces together, label this new shape *Shape 1*, and set it aside.



Step 2: Divide a second copy of the original circle into *eight* congruent wedges. Rearrange the parts as shown, tape the pieces together, label this new shape *Shape 2*, and set it aside.



Step 3: Repeat the process with a new circle, dividing it into 16 congruent wedges. Rearrange the parts in a similar way, and tape them together. Label this new shape *Shape 3*, and add it to your collection.



Note: *Shape 3* resembles *Shape 2* (shown on the previous page), but has 16 wedges rather than 8. It is formed by cutting, rearranging, and taping the wedges from the circle above.

The process you started can be thought of as the first three steps in a potentially endless process. Each step produces a shape that has two scalloped sides (the top and bottom in our pictures) and two straight sides.

9. Write and Reflect Here's your last chance to see where this is heading before the hints come! Look at your three shapes. It is not practical to continue the process by hand, but your *mind* can continue it. Think about what *Shape 10* might look like, and sketch (roughly) the picture you've made in your mind. Using *words* (no formula for now), explain how this diagram might lead to a way of finding the area of a circle.

What do *radius* and *circumference* mean?

10. Call the radius of your original circle r and the circumference C. Using those symbols, describe

a. the length of the scalloped sides of *Shape 3*;

b. the length of the straight sides of *Shape 3*.

11. Think about what changes and what does not change from step to step in the process.

Think about formal things like lengths and also about informal things like "rough appearance."

Scalloped-side invariants: List a few things that do *not* change about the scalloped sides from step to step in this process.

Scalloped-side changes: List things that *do* change about the scalloped sides.

Straight-side invariants: List things that do *not* change about the straight sides.

Straight-side changes: List things that *do* change about the straight sides.

Area: How does the area of each shape in your series compare to the area of the original circle?

Where necessary, explain the items in your lists to make them clear.

12. Mathematicians describe this process as producing a “sequence” of shapes that “approaches a rectangle.” Explain what “approaches a rectangle” might mean for this sequence.

- **a.** In what way do the shapes more and more resemble a parallelogram?
- **b.** Why a *rectangular* parallelogram?

13. Build a formula.

- **a.** If you carried the process far enough so that the new shape is “almost a rectangle,” what would be the approximate dimensions (base and height) of that rectangle in terms of r and C?
- **b.** Using the approximate dimensions you just found and the area formula for a rectangle, what is the approximate area of *this* “almost rectangle”?

See the *Connected Geometry* module *A Matter of Scale* for an in-depth study of similarity and more information about π.

The value of $\frac{C}{d}$ is often approximated as $\frac{22}{7}$, but no ratio of integers can specify the value of $\frac{C}{d}$ precisely. Similarly, the value 3.14 is often “good enough for practical purposes,” and 3.14159 is quite precise, but no decimal could have enough digits to show the *exact* value. That is one reason why the symbol π is used.

All circles are *similar* (they have the same shape). Therefore, the ratio of the circumference to the diameter in one circle is the same as the ratio of the circumference to the diameter in any other circle. This ratio is invariant: it doesn’t depend on the size of the circle. The ratio $\frac{C}{d}$ is named by the Greek letter π (“pi”). That is, π is *defined* to be the ratio of the circumference to the diameter of a circle:

$$\pi = \frac{C}{d} = \frac{C}{2r}.$$

14. Use the results of Problem 13 and the formula $C = 2\pi r$ to write a formula for the area of the “almost rectangle” in terms of r. (That is, eliminate the C in the formula.)

FOR DISCUSSION

Mathematical thinking *must* be precise. So what sense does it make for us to be talking about "almost rectangles" as if "almost" was just as good as the real thing?

If you can identify the gaps in an argument, you might be able to fill them immediately or later.

While your results actually should be correct, the thinking outlined above makes some questionable assumptions and does leave out some steps. Discuss the strengths and weaknesses of this argument as *you* see them.

What holes in the argument would have to be filled in order to make it completely convincing?

15. Write and Reflect You can measure a line segment with a ruler. How would you measure the circumference of a circle?

16. Write and Reflect Suppose you had *only* a ruler. Write an algorithm that would allow you to take measurements and to get as close as you wanted (within the limits of the ruler's accuracy) to the circumference of a circle.

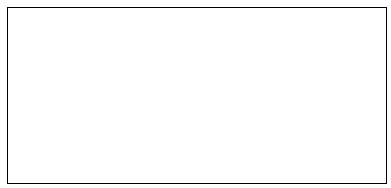
CHECKPOINT

17. To answer the following questions, refer to the seven shapes on the next page. Be sure to provide reasons for your answers.

- **a.** Find two shapes that have the same area. Explain how you decided their areas were the same.
- **b.** Group the shapes by area.
- **c.** Is the area of shape 1 greater than, less than, or equal to the area of shape 4?
- **d.** Compare the areas of shapes 1 and 3 as you did for shapes 1 and 4.
- **e.** Compare the areas of shapes 2 and 3.
- **f.** Compare the areas of shapes 2 and 5.
- **g.** Compare the areas of shapes 6 and 7.
- **h.** Compare the areas of shapes 2 and 7.

Investigation 3.4

AREA FORMULAS *(continued)*



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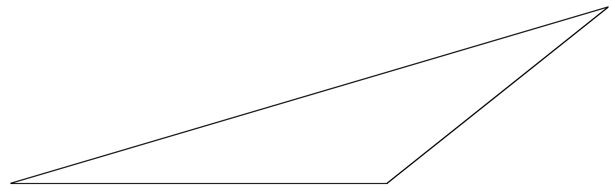

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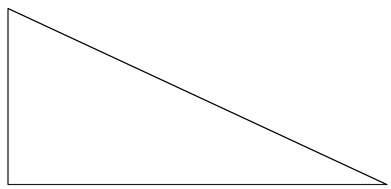

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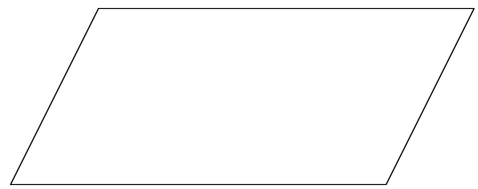

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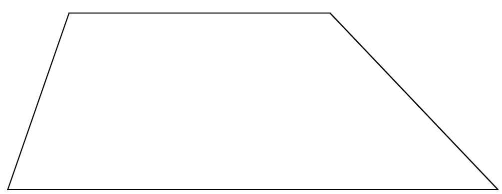

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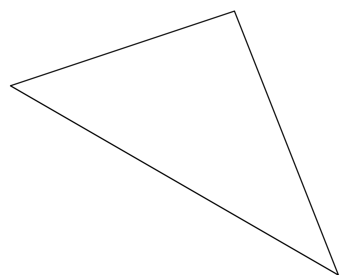

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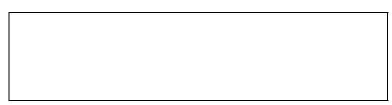

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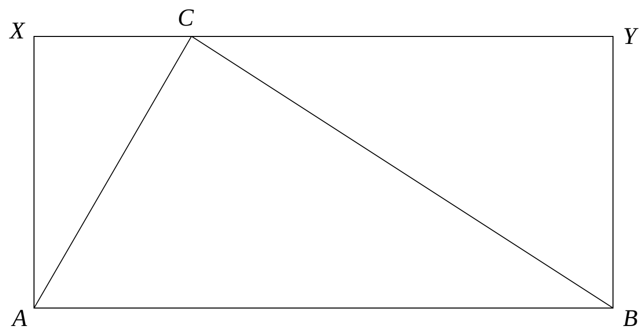
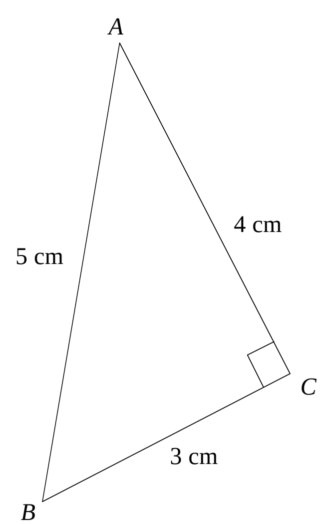
18. Is the sum of the areas of $\triangle ACX$ and $\triangle BCY$ in the rectangle below greater than, less than, or equal to the area of $\triangle ABC$? Justify your answer.



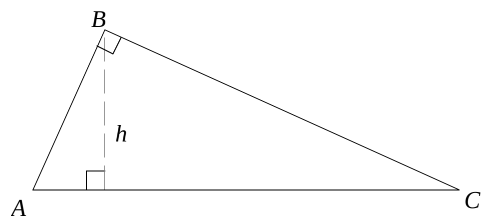
19. Below is a picture of right triangle.

a. What is the height from vertex A to base $\overline{BC}$?

b. What is the area of the triangle?



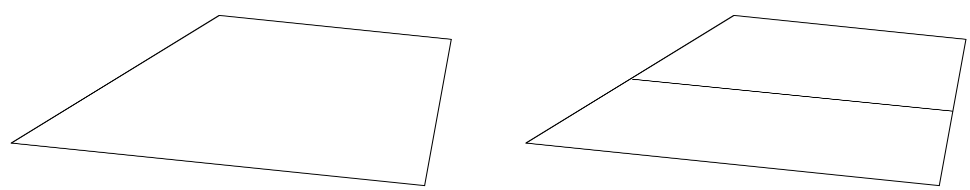
20. In the figure below, $m\angle ABC = 90^\circ$ and h is the altitude to base $\overline{AC}$ of $\triangle ABC$. Compare the quantities $AC \cdot h$ and $AB \cdot BC$. Explain.

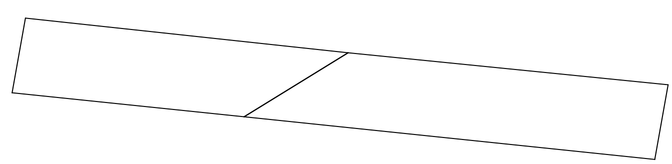


TAKE IT FURTHER. . . . .

UNIFYING THREE IDEAS

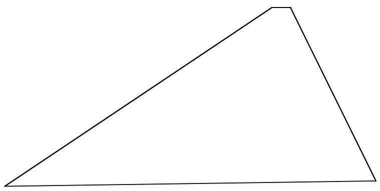
The following three problems ask you to think about how the area formulas for triangles, trapezoids, and parallelograms are related. Below is a series of pictures, showing one way a trapezoid can be dissected into a parallelogram.





21. **a.** Describe the dimensions of the parallelogram in terms of the dimensions of the original trapezoid.

b. Describe the area of the parallelogram in terms of the area of the original trapezoid.



Picture a trapezoid like this, only with an even smaller base on top: "infinitesimally small," a mathematician might say.

c. Derive an area formula for trapezoids from this relationship.

22. a. In your head, imagine redoing Problem 21 with a trapezoid in which one base is a "normal" size, but the other is so small that it can hardly be seen. What does such a situation tell you about the area formula for triangles?

b. What happens to the area formula for trapezoids if one of the bases is extremely small—practically a single point—compared to the other?

23. What happens to the area formula for trapezoids if both of the bases are the same size? What kind of figure *is* a "trapezoid with congruent bases"?

INTUITION AND THE AREA OF TRIANGLES

The next problem asks you to think *intuitively* about the area of triangles, without using formulas.

24. A Fact and a Challenge

Geometry software might help you to get a picture of this "dragging"—varying one part of a figure while keeping other parts the same.

The fact The area of the triangle may change when one of its vertices is moved (or dragged) around. But there are some directions to move the vertex so that the area of the triangle does not change.

The challenge Pick a vertex, and imagine moving it about (or actually move it about using geometry software). Do the following *without* measuring the area of the triangle.

a. Find directions of movement that you are *certain* will increase the area, and explain why you are certain they will. Do the same for directions that will decrease the area.

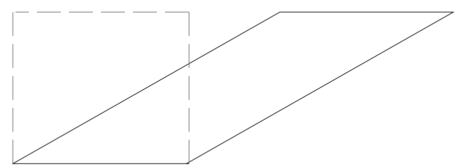
b. If there are directions in which to increase the area and directions in which to decrease it, there might be directions in which the area is invariant. Find a direction that seems likely to keep the area invariant, and explain why you picked it.

c. Check your direction in part b and correct it if necessary. Relate what you've seen to the area formula for a triangle.

EXTREME CASES

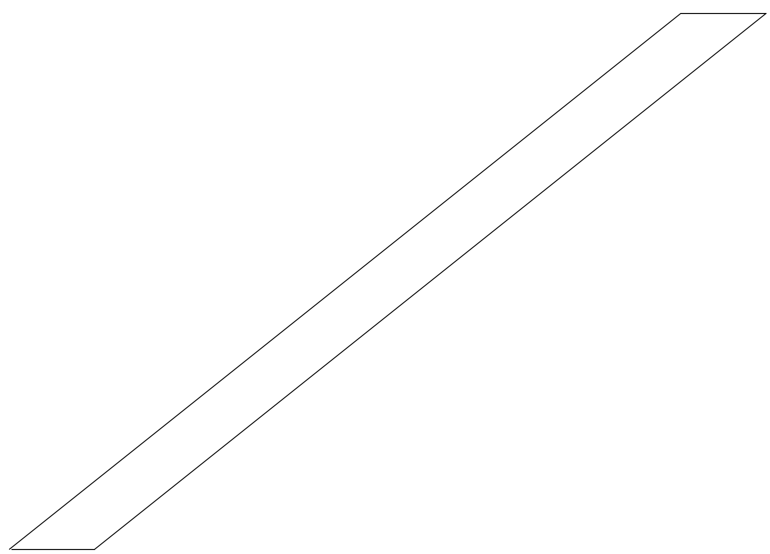
In previous problems, you have dissected parallelograms into rectangles without restrictions. What if, however, the rectangle must have a specific baselength? The following two problems address this problem.

25. Show that the parallelogram can be dissected and superimposed onto a rectangle with the same base and height (as shown). Trace the figures and cut them out to work with, or use geometry software.



26. The parallelogram below is an extreme example of the one in Problem 25. Trace and copy it; then figure out how to cut it up and rearrange the pieces into a rectangle with one side equal to the *smaller* side of the parallelogram.

Many students find it helpful to draw a rectangle with the appropriate base and height and then try to fill it in with pieces of the parallelogram.



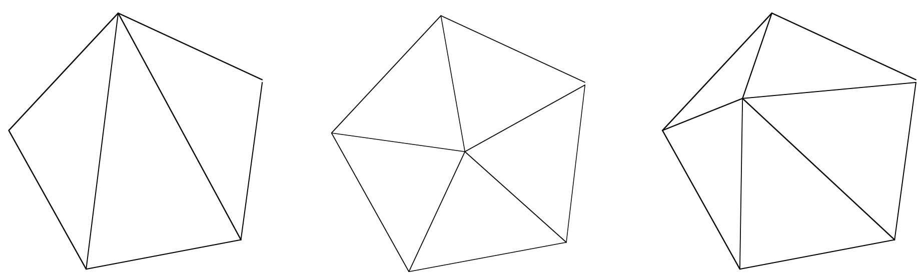
REGULAR POLYGONS AND CIRCLES

In this module, *triangulation* means cutting shapes up into triangles. Since lots is known about triangles, this is often a useful way to discover properties of other shapes, including area formulas.

27. Can all shapes be triangulated with a finite number of cuts? All polygons? Try to make a polygon that *can't* be triangulated.

Remember that in a regular polygon, all sides are congruent (same length) and all angles are congruent (same measure).

It's pretty clear that all *regular* polygons can be triangulated; in fact, you can do it in more than one way.



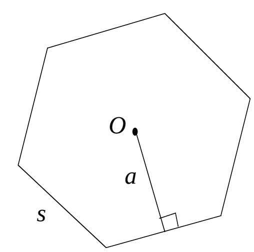
Logo is ideal for this. In fact, if you used the module *Habits of Mind* in your class, you may already have a program that does just this. You can simply run it for each number of sides and print the result.

28. Draw a series of regular polygons; start with four sides and stop when your shape starts to look like a circle.

29. Triangulate each of your polygons in the same way: start from the center point and draw segments to each of the vertices. How many triangles are in each polygon? What can you say about these triangles and their areas?

30. Use the following to write an area formula for regular polygons:

Call the distance from the center point of a regular polygon to a side of the polygon a.



(The letter a is used here because the mathematical term for this distance is the *apothem*. It's also the height of the triangles.)

In your formula, you may want to use some of the following variables: a for the apothem, n for the number of sides, s for the length of a side, and p for the perimeter of the polygon. Be sure to explain how you came up with the formula.

31. Write and Reflect In Problem 30, you assumed that there was a "center point" in every regular polygon.

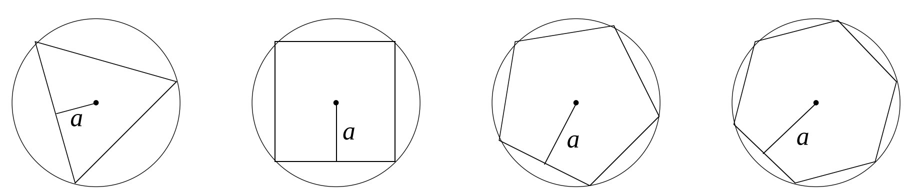
a. Write a definition of center point and explain why one has to exist in every regular polygon.

b. Give an algorithm for locating the center point of a regular polygon.

c. What properties of the center point did you use in Problem 30?

d. Can nonregular polygons have a center point? Explain.

32. Look at the series of regular polygons below. They are all inscribed in the same circle.



As the number of sides increases, how does the perimeter of the polygon compare with the circumference of the circle?

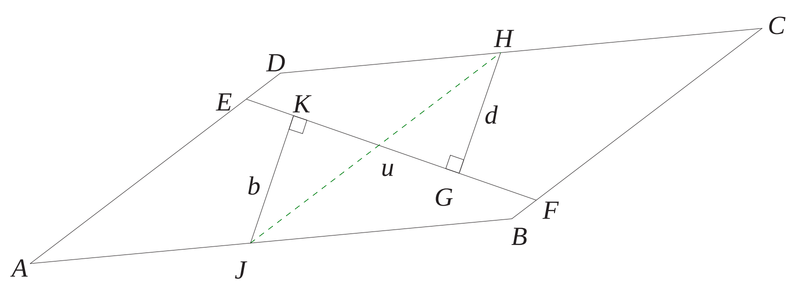
33. As the number of sides increases, what does the length a approximate?

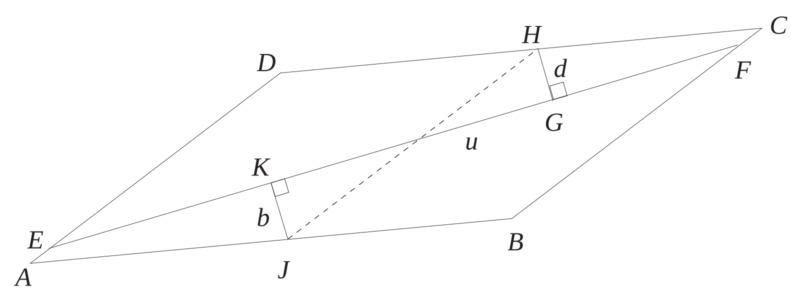
34. Give a convincing argument to extend your area formula for regular polygons into an area formula for circles.

DIMENSIONS

When you dissected a parallelogram into a rectangle, you probably preserved the base and height. Here is an algorithm for dissecting a parallelogram (which might, itself, be a rectangle) into a rectangle that has a *different* base, one that you might choose. To use this algorithm, the base for the resulting rectangle must be no longer than the longest diagonal of the given parallelogram.

Step 1: Draw $\overline{EF}$ (the segment you want for a base of your rectangle) from one side of the parallelogram to the opposite side. If the sides of parallelogram $ABCD$ are not all congruent, this first segment must connect the *shorter* sides ($\overline{AD}$ and $\overline{BC}$ in the pictures below).





The illustrations show a "long" and a "short" choice for $\overline{EF}$.

Step 2: Crossing roughly in the middle of $\overline{EF}$ (getting the precise middle is rarely important), draw $\overline{JH}$ parallel to $\overline{AD}$ and $\overline{BC}$.

Step 3: From H and J, draw $\overline{HG}$ and $\overline{JK}$ perpendicular to $\overline{EF}$.

Step 4: Cut along $\overline{EF}$, $\overline{JK}$, and $\overline{HG}$.

Step 5: Rearrange the four parts into a rectangle with base EF and height $JK + GH$. (This algorithm omits instructions on *how* to do the rearrangement. That is left for you to figure out!)

35. **Project** Investigate the algorithm described above. Here are some questions you might pursue: Why does it work? Is it foolproof (that is, does it work in all cases)? Why is Step 2 important? Why, then, is the “precise middle” in Step 2 “rarely important”? (When *is* it important?) Is there a way to adapt the process if you wanted to end with a rectangle longer than the parallelogram’s longest diagonal? How could you choose your first cut in order to end with a rectangle that is approximately square? Can you simplify the algorithm?

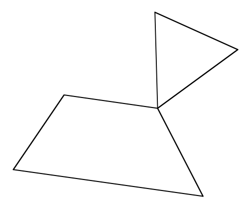
Investigation 3.5

More on the Midline Theorem

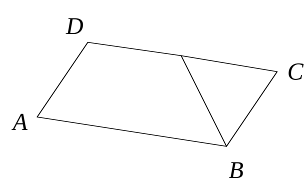
One student used the following method to dissect a triangle into a parallelogram:

Start with your triangle and cut between the midpoints of two sides.

Rotate the top triangle around one of the midpoints. The two segments will match because you cut at a midpoint.



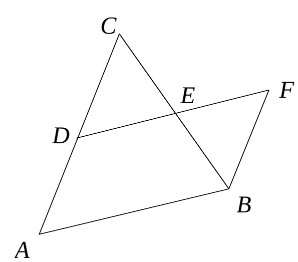
ABCD is a parallelogram because the opposite sides are congruent. $\overline{AD} \cong \overline{BC}$ because they were made by cutting at a midpoint. And $\overline{AB} \cong \overline{CD}$ because a midline cut makes a segment half as long as the base.



This student made use of an idea that you may or may not have encountered: a midline cut. The student claims that cutting along the midline of a triangle produces a segment that is half as long as the base. Does that seem reasonable?

1. **Write and Reflect** Explore the midline cut. You can do this by cutting triangles and comparing lengths, by constructing triangles and taking measurements, or any other way that you like. The idea is to do enough so that you either definitely believe or definitely don't believe the student's statement about the midline cut. When you're done, write a few paragraphs to explain how you investigated the problem and what you decided.

If the student's statement about the midline cut seems reasonable, the next step should be to try and prove it. (If it is not reasonable, the student needs an entirely new argument!) You have all the tools necessary for such a proof: ideas about creating and writing a proof, information about triangle congruence, and knowledge of the properties of parallelograms. The next few problems will help you construct a proof. Your job at the end of the problem set will be to write up a formal proof of the midline cut conjecture (which you can then call the Midline Theorem), using one of the methods of proof presentation that you learned about earlier.



To prove that the midline cut works, we need a slightly different construction. In $\triangle ABC$ above,

- D is the midpoint of $\overline{AC}$.
- E is the midpoint of $\overline{BC}$.
- D, E, and F are collinear.
- $\overline{DE} \cong \overline{EF}$.

2. Prove that $\triangle DEC \cong \triangle FEB$.

3. Show that, if $\triangle DEC \cong \triangle FEB$, $ABFD$ is a parallelogram. Don't use any information about the length of segments $\overline{DE}$ or $\overline{DF}$.

You can answer this question, even if you haven't solved Problem 3 yet. Just pretend you have solved it.

4. If *ABFD* is a parallelogram, list everything you can conclude about $\overline{DE}$ and $\overline{AB}$.

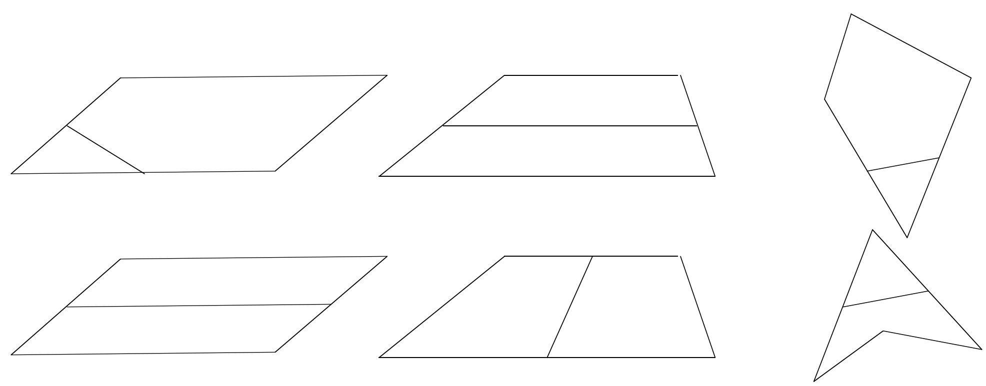
5. Write a proof of this statement, using a style of proof that you have learned.

THEOREM 3.1 *The Midline Theorem*

In a triangle, a midline is parallel to the third side and half as long.

TAKE IT FURTHER. . . . .

The idea of a midline—a segment joining two midpoints in a triangle—can be generalized to quadrilaterals. Here are a few pictures of possible midlines for quadrilaterals:



6. Experiment with the various possible meanings for "midline" of a quadrilateral. Can you find any relationship between a midline and the sides of a quadrilateral? Between a midline and a diagonal? Answer the following questions:

- How did you define "midline" for a quadrilateral? (Joining any two midpoints? Two opposite midpoints? Two adjacent midpoints?)
- With what kinds of quadrilaterals did you experiment?
- What did you find? Are there any special properties of a midline of a quadrilateral? Can you make any conjectures?

Investigation 3.6

THE PYTHAGOREAN THEOREM

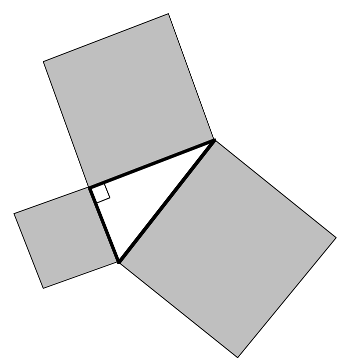
Reasoning about dissection and rearrangement has been useful throughout mathematics. Many proofs of the Pythagorean Theorem, a famous and valuable fact about right triangles, use dissection. In Euclid's classic text *The Elements*, the theorem is stated this way:

THEOREM 3.2 ***The Pythagorean Theorem***

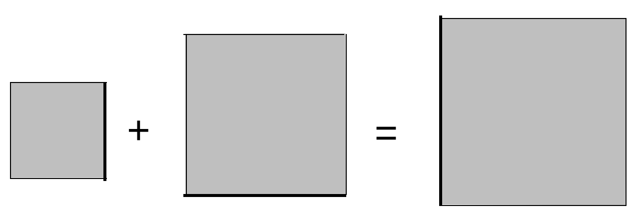
In a right triangle, the square built on the hypotenuse is equal to the squares built on the other two sides.

PERSPECTIVE ON THE PYTHAGOREAN THEOREM

Euclid viewed the Pythagorean Theorem differently from the way we view it today. This essay will show you some of the ways conceptions of area and length have changed over the years.

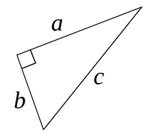


As it is worded, the theorem is about the relationship among three squares, and that is how Euclid meant it.



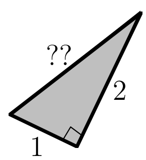
Today, most people think of the theorem as stating a relationship among three numbers, a, b, and c, which represent the lengths of the sides of a right triangle.

If c is the length of the hypotenuse, then c^2 is the area of the square on that hypotenuse. The theorem states that (the area of) *that* square is the same as (the combined area of) the other two squares: $c^2 = a^2 + b^2$.

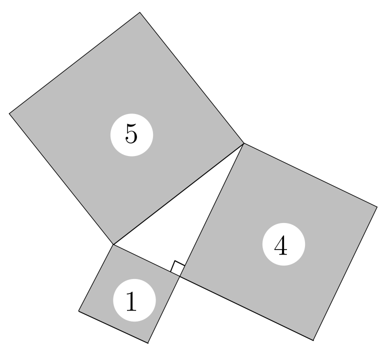


But things were different in Euclid's day. People could compare two areas—using terms like equal, less, twice, or in the same proportion as—without necessarily being able to give a numerical value for either area.

The same was true of lengths. People could draw a right triangle that had legs of lengths 1 and 2, but they had not yet conceived of a kind of number to describe the length of the hypotenuse. (That number is now known as $\sqrt{5}$). They *knew* that the length could not be any of the numbers they knew about. They also managed to prove that, whatever the length was, it wasn't even a ratio of the lengths for which they had numbers. So, for all they could say about the length, they could not give it a numerical value.



The areas of two of the squares were easy to compute from the lengths of their sides. The Pythagorean Theorem told them that the third area—which could not be computed in the same way because its sidelength was unknown—could be computed as the sum of the other two. Thus, the original Pythagorean Theorem was not about lengths, but only about areas.



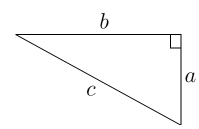
Over the years, thousands of proofs of the Pythagorean Theorem have been discovered. In this investigation, you will explore a proof that uses dissection and area.

A DISSECTION PROOF

The proof outlined below is probably from China, from about 200 B.C. Most likely, the author of the proof developed the theorem independently, rather than learning of it from the Pythagoreans.

For the proof outline, follow the directions at each step, and answer the questions as you work. When you are finished, you will have constructed a proof of the theorem!

Step 1 Construct an arbitrary right triangle that is not isosceles. Label the short leg a, the long leg b, and the hypotenuse c.

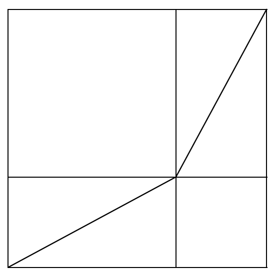


(Note: This proof will still work if the triangle is isosceles; then $a = b$. In this case, we cannot talk about a "short leg" and a "long leg.")

Step 2 Construct two squares whose sides have length $a + b$.

Step 3 Dissect one of the squares as shown below:

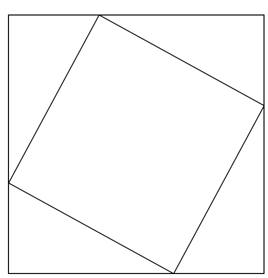
- a square with sidelength a in one corner;
- a square with sidelength b in the opposite corner;
- two rectangles cut at their diagonals into four triangles.



1. Show that each of the four triangles you've just created is congruent to the original right triangle.

Step 4 Dissect the other square into five pieces as shown in the picture below:

- four triangles congruent to the original right triangle;
- a remaining piece in the center.



2. Show that the piece in the center is a square with sidelength c.

- Explain why all of its sides have length c.
- Explain why all of its angles are right angles.

Step 5 The two original squares have the same area.

Area of this square *equals* *area of this square.*

The eight triangles are congruent, so the four from the first square are equal in area to the four from the second square.

Shaded area here *equals* *shaded area here.*

Step 6 Remove the four triangles from each square. What remains in the first square will have the same area as what remains in the second square.

Unshaded area here *equals* *unshaded area here.*

The *geometric* equality—the Pythagorean Theorem as Euclid knew it—has been demonstrated.

Our more modern algebraic interpretation

$$a^2 + b^2 = c^2$$

follows from the algebraic formulas for the areas of squares. The areas of the two squares on the left are a^2 and b^2; the area of the square on the right is c^2. Geometric reasoning tells us that the areas on the left (a^2+b^2) and right (c^2) are equal: $a^2+b^2 = c^2$.

3. **Write and Reflect** Test your own understanding. Put away these pages, and see if you can write this entire proof from memory. Don't worry about the wording as long as your writing is clear and the reasoning is complete.

Perspective on the Pythagoreans

Do you know who the Pythagorean Theorem is named for? This essay will introduce you to Pythagoras and the secret Pythagorean Society of ancient Greece.

The Pythagorean Theorem is named for one of the most famous of the ancient mathematicians, the Greek Pythagoras, who lived about 569–500 B.C., about the same time as Lao-Tse, Buddha, and Confucius. Pythagoras spent much of his early life traveling. During that time he studied with Thales, the so-called "Father of Geometry." Thales was known for insisting on proofs rather than accepting intuition about geometric ideas, and his student Pythagoras also took proofs very seriously.

After his travels, Pythagoras settled in Italy (then a part of the Greek Empire) and started a secret society known today as the Pythagoreans. The Pythagoreans met to study mathematics, music, and astronomy. There were religious and political aspects to their Society as well. For example, the Pythagoreans believed that, when a person died, the soul moved into another body. Like many religions, they also had special rules around foods: for them, eating beans and drinking wine were prohibited.

Each member of the Society was sworn to secrecy, and all mathematical discoveries were attributed to Pythagoras. It is possible that Pythagoras discovered his famous theorem on his own, or perhaps he learned of it during his early travels and merely discovered a new proof. Or it could be that one of his students in the Society discovered the theorem. The Pythagoreans valued women and their thinking. Theano, the wife of Pythagoras, was a mathematician in her own right.

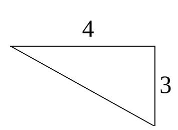
In any case, the theorem was known to others before the Pythagoreans discovered it.

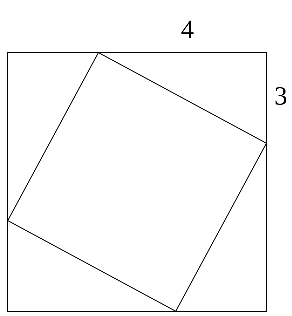
Babylonians knew the theorem around 1500 B.C., nearly 900 years before the birth of Pythagoras. Mathematicians in other cultures had probably discovered it as well. Ancient Egyptians used the related fact that a triangle with sides of 3, 4, and 5 units is a right triangle to mark off right angles for property boundaries. Mathematicians in ancient India discovered several of what are now called *Pythagorean triples*, sets of three integers that satisfy the relationship $a^2 + b^2 = c^2$.

The vow of secrecy and the Pythagorean Theorem may have caused the death of at least one member of the Society of Pythagoreans. Pythagoras and his students believed that everything in the universe depended on whole numbers. They believed that the number 1 was divine because it was the building block for all other numbers. Rational numbers were acceptable to the Society because they represent a *ratio* of two whole numbers. Irrational numbers, however, went against their religion in a fundamental way. The discovery that $\sqrt{2}$ is not rational remained a religious secret until Hippasus, a member of the Society, disclosed the secret to outsiders. It is said that members of the Society drowned him in the sea as punishment.

The Pythagoreans' political activity—at one time they held power in several cities in southern Italy—worried the government, and eventually the Society was forced to disband. Pythagoras died in exile.

USING THE THEOREM





4. Verify the Pythagorean Theorem numerically by testing a specific case:

a. Construct a right triangle with one leg 3 inches long and the other leg 4 inches long (the picture here is not drawn to scale). What is the area of your triangle?

b. Construct a square with sides of length of $3 + 4 = 7$ inches. What is the area of your square?

c. Dissect the square into five pieces as shown on the left:
- four right triangles congruent to the original one;
- one square in the middle.

Find the area of the square in the middle by subtracting the areas of the four right triangles.

d. Calculate $a^2 + b^2 = 3^2 + 4^2$. Is the sum equal to the area of the middle square?

5. Try Problem 4 again, but start with a triangle whose legs are 5 centimeters and 12 centimeters long.

If the area of a square is 4, how long are its sides?

6. How long is one side of the middle square in Problem 4?

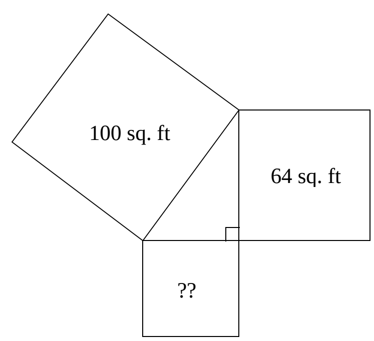
7. Draw a right triangle with legs 5 cm and 12 cm. Draw a square whose side is the hypotenuse of this triangle.

a. Use the Pythagorean Theorem to find this square's area.

How long is *one* side?

b. What is the square's perimeter?

8. The picture below shows squares on the sides of a right triangle, and gives the area of two of the squares.



a. Find the missing area.

b. Find the lengths of the three sides of the triangle.

This triangle is not easy to construct; you'll have to be a little creative.

9. Construct a right triangle with a 17-cm hypotenuse and a leg of length 8 cm. Draw a square on the *other* leg of the triangle.

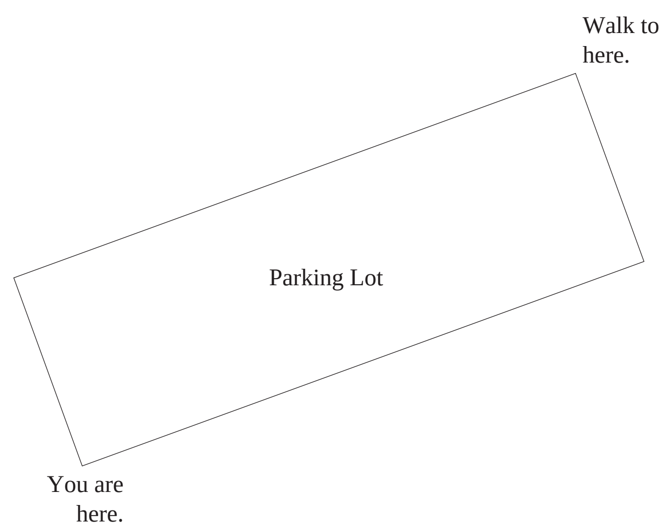
a. What is the area of the square you've drawn?

b. What is the length of the other leg of the triangle?

By reasoning about squares on the sides of a right triangle, and by using the area formula for a square, you can find the length of any side of a right triangle if you know the other two sides. But how often does one need to know the sides of a right

triangle? Remarkably, the Pythagorean Theorem is one of the most widely-applied facts in mathematics and throughout our daily lives. Below are a few examples of its use.

10. **Distance** You are standing at one corner of a rectangular parking lot measuring 100 feet by 300 feet.



a. If you walked along the sides of the parking lot to the opposite corner, how far would you walk?

b. If you walked diagonally across the parking lot to the opposite corner, how far would *that* be? How much shorter or longer is it than walking along the sides?

c. Of course, there might actually be cars in the parking lot, blocking you from walking directly on the diagonal. In this case, how might the two paths—walking along the sides of the whole parking lot, or zigzagging through the lot—differ in length? Explain.

11. Find the length of the diagonal of a square whose sides have length

a. one foot;

b. two feet;

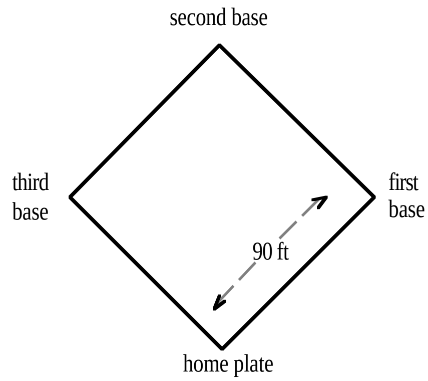
c. four feet;

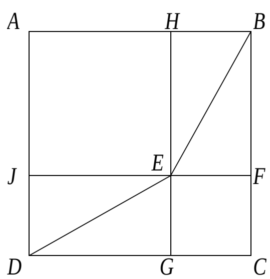
d. ten feet;

e. 100 feet.

12. Find a pattern in the lengths you calculated in Problem 11. Write a simple rule relating the diagonal of a square to its sides.

13. A baseball "diamond" is really a square 90 feet on a side. How far is second base from home plate?





14. In this familiar configuration, the large square $ABCD$ has two smaller squares drawn inside it, $\square CGEF$ and $\square AHEJ$. If $CF = 1$ and $BF = 3$, find the following lengths and areas:

a. AB

b. BE

c. BJ (not drawn)

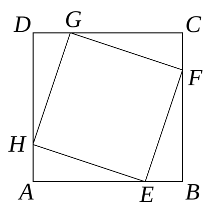
d. HJ (not drawn)

e. CE (not drawn)

f. area of $AHEJ$

g. area of *CGEF*

h. area of $\triangle BEF$.

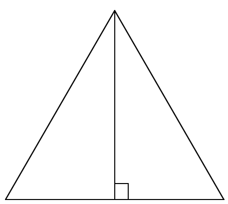


15. You have seen *this* picture before, too. If $AE = BF = CG = DH = 3$, and $EB = FC = GD = HA = 1$, find:

a. *EF*

b. perimeter of *ABCD*

c. area of *ABCD*.



16. Find the height of an equilateral triangle with sides of length

a. one cm;

b. two cm;

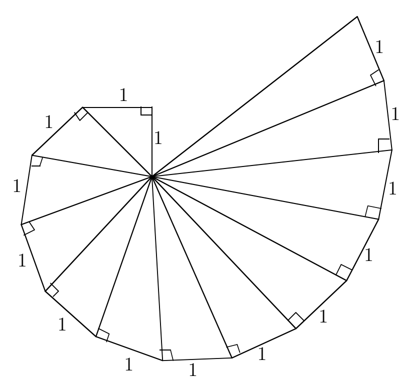
c. three cm;

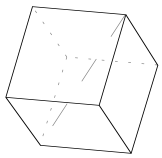
d. ten cm;

e. 100 cm.

17. Find a pattern in the lengths you found in Problem 16, and write a rule relating the altitude of an equilateral triangle to the sides of the triangle.

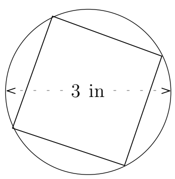
18. Find the lengths of all the segments that are not labeled in the picture below. Describe a pattern in the lengths.





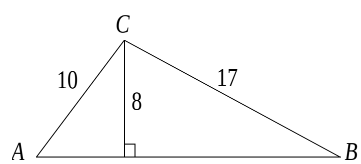
19. Here is a picture of a cube with one of its diagonals. If the edges of the cube are 10 inches long, how long is the diagonal?

20. An airplane left Los Angeles. After flying 100 miles north, it turned due east and flew 600 miles, then turned north again and flew 350 miles. About how far was the airplane from its starting point?
(**Challenge** Why does the Pythagorean Theorem *not* give the precise distance here?)

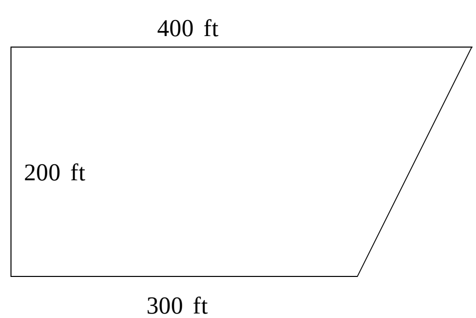


21. A square is inscribed in a circle with a 3-inch diameter. What is the area of the square?

22. Triangle *ABC* below is *not* a right triangle. Find its area.



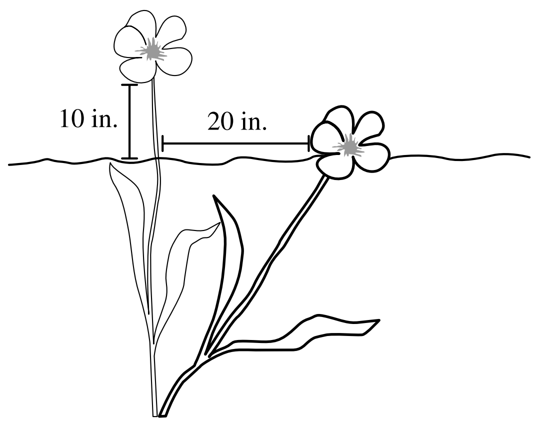
23. What is the area of the plot of land shown in the diagram? (Explain any assumptions you make.)



TAKE IT FURTHER

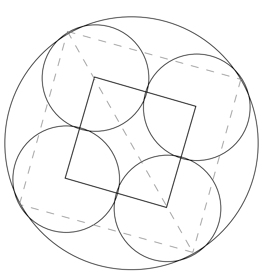
24. A water lily is growing in a pond, rooted to the pond floor. A blossom is 10 inches above the water when it's standing. When it's pulled 20 inches to the side, the blossom just touches the surface of the water. How deep is the pond?

There *must* be a better way of measuring depth than this!



25. Write and Reflect Find out more about Pythagoras and the secret Society of Pythagoreans. Besides the theorem that bears his name, Pythagoras and his students made important discoveries in the areas listed below. Choose one of these topics to research. Find out as much as you can about it, and write a short paper to explain your findings to the class.

- Irrational numbers, and the proof that $\sqrt{2}$ is irrational
- The golden ratio
- The five regular (Platonic) solids
- Intervals on the musical scale



26. A Packing Problem (Challenge) Four wires run through a cable. In the cross section of the cable shown at the left, the wires are tangent congruent circles centered on the vertices of a square. The outside shield of the cable is the large circle, which *circumscribes* the other four. The dashed lines suggest some

Tangent circles **touch at only one point.**

relationships that you need to know, but proofs of these relationships might not be obvious. For this problem, assume them without proof.

a. If the small circles have a diameter of 1 mm, what is the diameter of the large circle?

b. Why is this *not* the same question as the following: "If the wires are 1 mm wide, how wide is the cable?"

27. The ratio of width to height for a television screen is required to be 4:3. The length of a diagonal of the screen is used to measure the advertised size. If you buy a 50-inch TV, the screen has a 50-inch diagonal. What are the width and height of this screen? What is its viewing area?

PYTHAGOREAN TRIPLES

A Pythagorean triple is a set of three positive integers, such as (3, 4, 5), which satisfy the equation $a^2 + b^2 = c^2$. If you have a Pythagorean triple, you can build a right triangle with sides of lengths a, b, and c.

28. Look back through your work in this lesson, and find two more Pythagorean triples.

29. The following triples are members of a *family* of Pythagorean triples. (There are other Pythagorean triples that do not belong to this family.) Check that each triple listed below *is* a Pythagorean triple. In what way are *these* triples enough alike one another to warrant calling them a family?

- (3, 4, 5)
- (6, 8, 10)
- (30, 40, 50)
- (45, 60, 75)
- (300, 400, 500)

30. Draw triangles with the following sidelengths. What do the triangles have in common?

- 6 cm, 8 cm, 10 cm
- 3 in., 4 in., 5 in.
- 15 cm, 20 cm, 25 cm

31. Find a Pythagorean triple that is *not* part of the (3, 4, 5) family. List four of its family members.

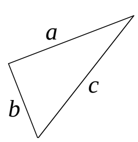
32. **Challenge** Find a way to generate Pythagorean triples.

CONVERSE OF THE PYTHAGOREAN THEOREM

To form the converse of a statement, switch the *if* and *then* parts. The converse of a true statement is not always true. Consider this example:

***Statement:* If you live in Los Angeles, then you live in California.**

***Converse:* If you live in California, then you live in Los Angeles.**



The Pythagorean Theorem could be stated this way: "If $\triangle ABC$ is a right triangle, then the sum of the squares on the legs equals the square on the hypotenuse." The *converse* of this theorem would be: "If for $\triangle ABC$ the sum of the squares on two sides equals the square on the third side, then $\triangle ABC$ is a right triangle."

This converse of the Pythagorean Theorem is actually what the Egyptians were using when they marked off right angles with (3, 4, 5) triangles. You have the tools you need to prove the converse of the Pythagorean Theorem, using what you know about triangle congruence.

33. An outline for a proof of the converse of the Pythagorean Theorem is below. You need to fill in the gaps to complete the proof.

Given: $\triangle ABC$, where the lengths of the sides satisfy $a^2 + b^2 = c^2$.

Prove: $\triangle ABC$ is a right triangle.

a. Construct a new right triangle with legs of lengths a and b.

b. What is the length of the hypotenuse of your new triangle? Why?

c. Your new triangle and triangle ABC must be congruent. Why?

d. $\triangle ABC$ must be a right triangle. Why?

34. Use the outline and your answers for Problem 33 to write a formal proof that if a triangle has sides whose lengths satisfy the equation $a^2 + b^2 = c^2$, then the triangle is a right triangle. Use one of the styles you learned about while studying congruence.

DISTANCE FORMULA

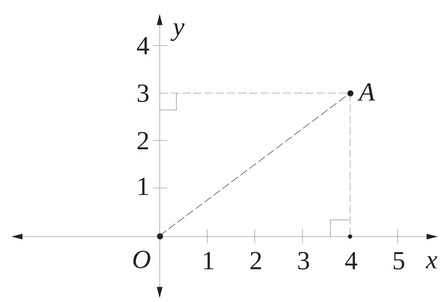
The distance between two points is the same as the length of the line segment connecting the two points.

Many geometry books teach a special formula for finding the distance between two points whose coordinates you know. That "distance formula" is just the Pythagorean Theorem in disguise. In the next several problems, you will discover the formula for yourself. By remembering that it's really just the Pythagorean Theorem, you can easily recall it.

35. Look at the picture below.

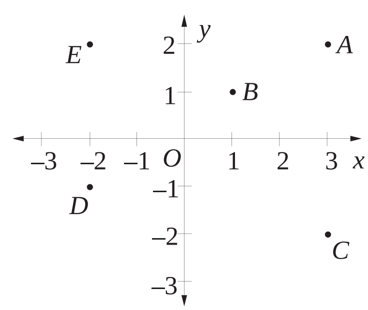
a. What are the coordinates of point A? How long are the sides of the right triangles in the picture?

b. How far is point A from point (0, 0), the origin?



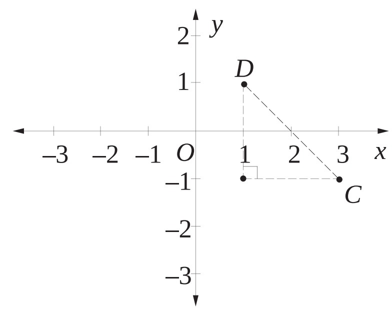
In problems like these, it may be helpful to copy the picture and sketch in the right triangles.

36. How far is each labeled point from the origin?

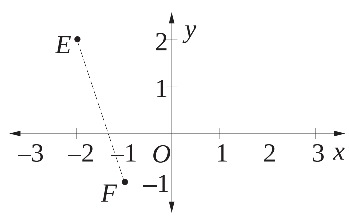


What are the coordinates of points *C* and *D*? How long is each side of the right triangle in the picture?

37. What is the distance between points *C* and *D*?

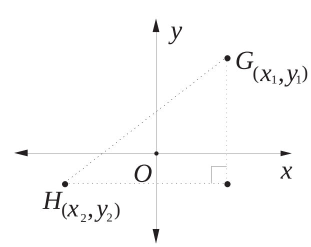


38. What is the distance between points *E* and *F*?

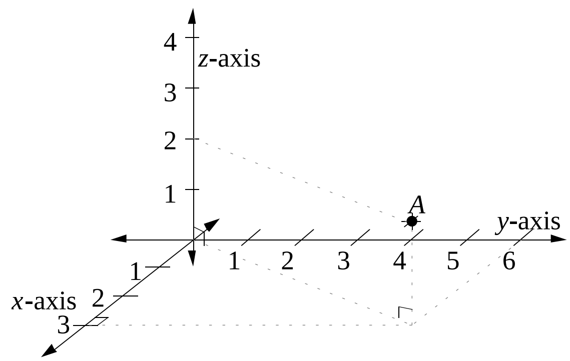


You might want to check to make sure your formula works in cases where the two points have the same *x*-coordinate or the same *y*-coordinate. In both of these cases, you can't draw a right triangle to help you find the distance. Do you still get a reasonable answer?

39. In the picture below, points *G* and *H* have variables for coordinates. The coordinates are given as (x_1, y_1) for *G* and (x_2, y_2) for *H*. Use these coordinates and the right triangle drawn in the picture to help you come up with a distance formula. This formula should work so that, if you know the coordinates of *any* two points, you can find the distance between them from the formula.



40. In three dimensions, points are given three coordinates, (x, y, z). How far from the origin is point A whose coordinates are (3, 6, 2)?



41. Try to extend your distance formula to a formula that works in three dimensions.

Investigation 3.7

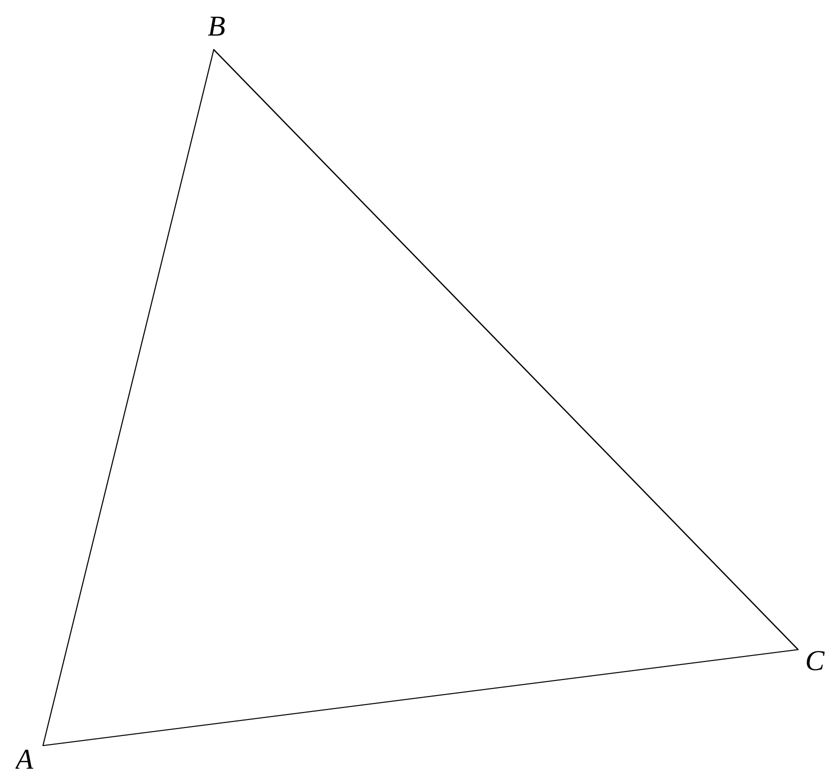
CHANGING SHAPE

Not only can you dissect a triangle into a rectangle, and vice versa, but you can dissect a triangle into a different triangle, or change a rectangle into a different rectangle. Now you will try some similar problems—problems that ask you to dissect given shapes into new ones with particular characteristics, or to cut a shape into parts that have particular characteristics.

MAKE A NEW TRIANGLE

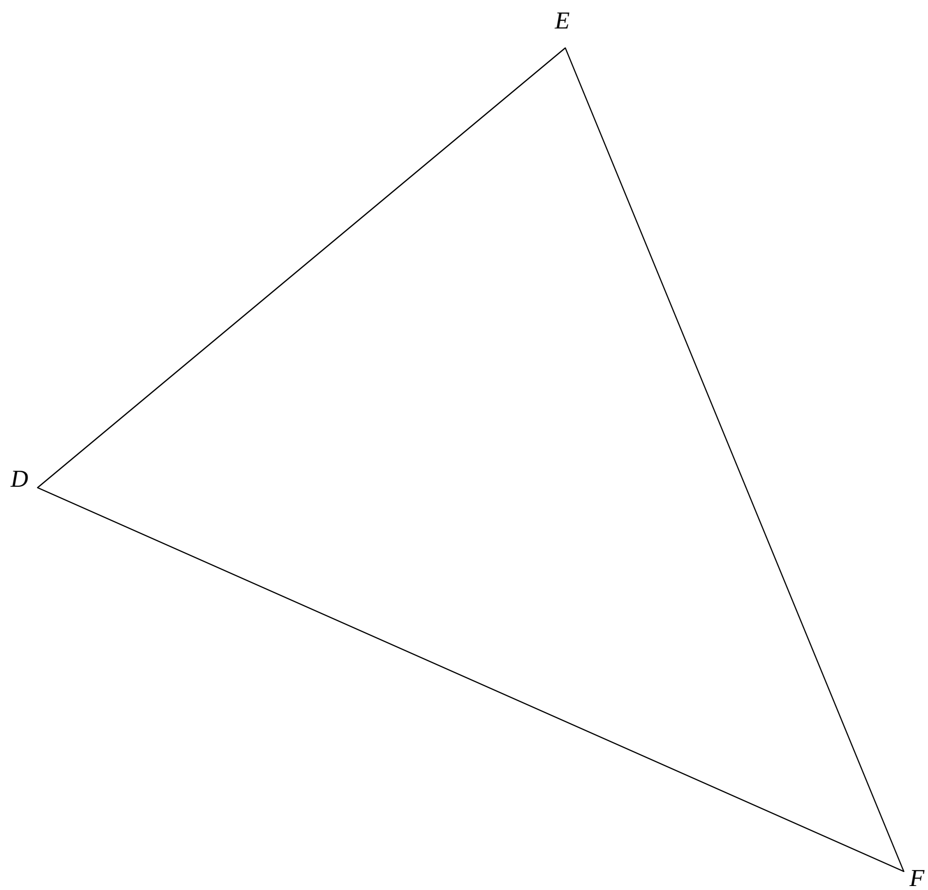
1. **Changing All the Angles** Find a way to dissect $\triangle ABC$ into *two pieces* that you can rearrange to form a new triangle (of equal area) with no angles congruent to any of the angles of $\triangle ABC$. Outline a presentation that will persuade the class that your dissection works.

To be persuasive, you should show three things: the final figure is a triangle (it has exactly three sides); the two joined edges match; and all three angles differ from those of the original triangle.



2. Using what you know about the area formula for triangles, think of a way to draw (by hand or with geometry software) another triangle of equal area that has no angles congruent to the angles of $\triangle ABC$.

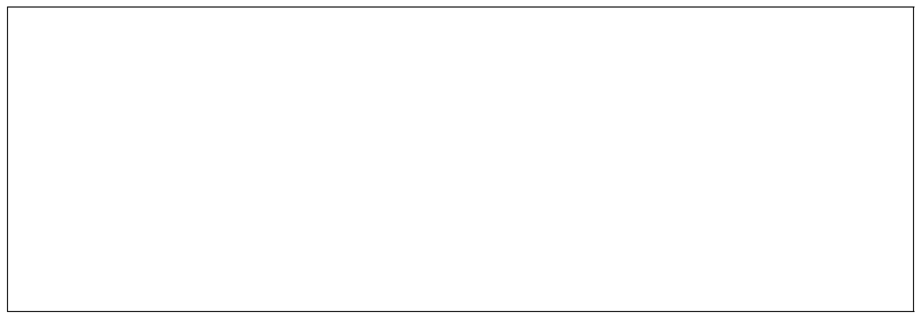
3. **Changing All the Sides** Find a way of dissecting $\triangle DEF$ into a triangle of equal area with no sides congruent to the sides of $\triangle DEF$. Justify your method.



4. Use a method other than cutting and rearranging to create a triangle of equal area with no sides congruent to the sides of $\triangle DEF$.

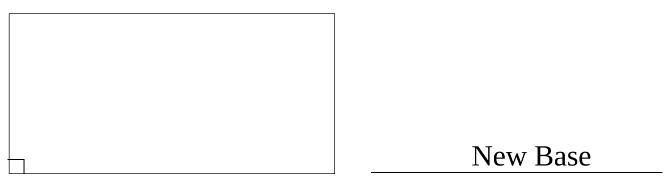
MAKE A NEW RECTANGLE

5. From the rectangle below, make a same-area rectangle with no sides congruent to the sides of the original figure.

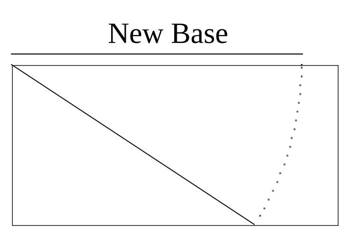


It may seem surprising, but a rectangle can be dissected into *any* other same-area rectangle. Given a starting rectangle and the base of the new one, there is a reliable way to construct the new rectangle. The method below usually works. (You'll fix a few bugs later.)

Step 1 Start with your rectangle with one side horizontal and the new base you want.

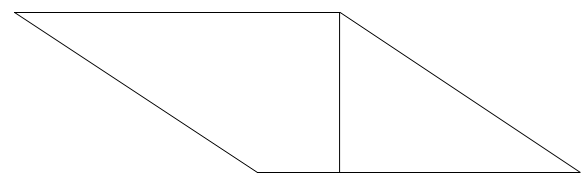


Step 2 From the upper-left corner of the rectangle, mark the point where the new base would hit the lower base of the rectangle. (A compass is useful for this.) Draw the segment, and cut along it.

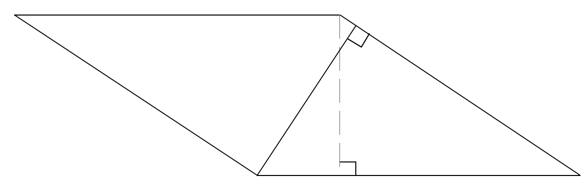


This parallelogram is important; one pair of sides is congruent to the base of the old rectangle, and the other pair is congruent to the base of the rectangle you want. Which is which?

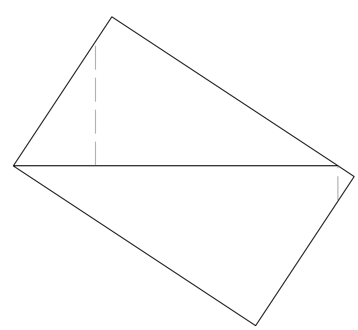
Step 3 Slide the triangular piece to the right, along the horizontal base of the rectangle, until the two vertical edges match up to create a parallelogram. (It may help to tape the new seam.)



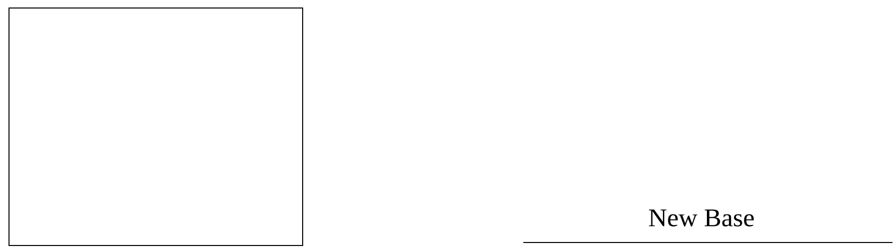
Step 4 From the bottom left vertex of the parallelogram, draw a line perpendicular to the two new slant sides. Cut along this perpendicular.



Step 5 Slide the triangular (bottom right) piece "north-west," parallel to the slant side of the parallelogram until the horizontal sides match up. You now have a rectangle with the desired base.

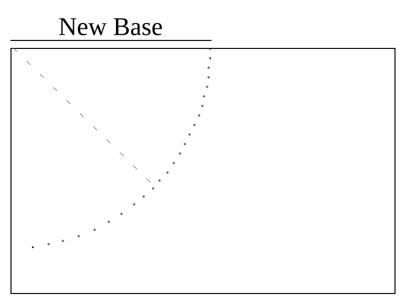


6. Try the algorithm: copy this rectangle and dissect it into another rectangle with the given base.

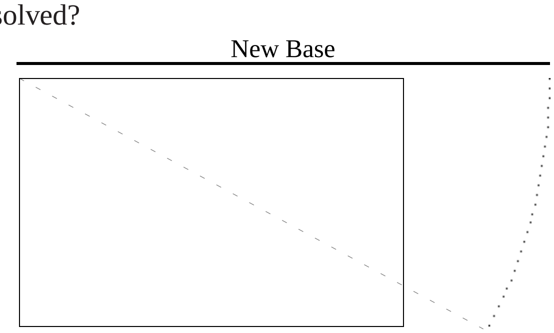


There are some minor bugs in the algorithm, but they can be fixed. The next few problems show other cases that reveal bugs in the algorithm. Alter the algorithm so that it will still work.

7. The required new base may be too short to reach the base of the original rectangle in Step 2. What additional work must you do to salvage the algorithm? Trace the rectangle in this picture, and turn it into a rectangle of the same area on the new base.

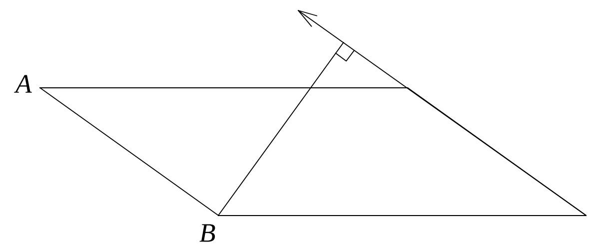


8. Step 2 of the algorithm also fails if the new base is too long. How can *this* problem be solved?



What starting conditions cause *this* to occur?

9. The perpendicular drawn in Step 4 could fall partly outside the figure (see below). Still, a rectangle of base $\overline{AB}$ must be built from this parallelogram. How?



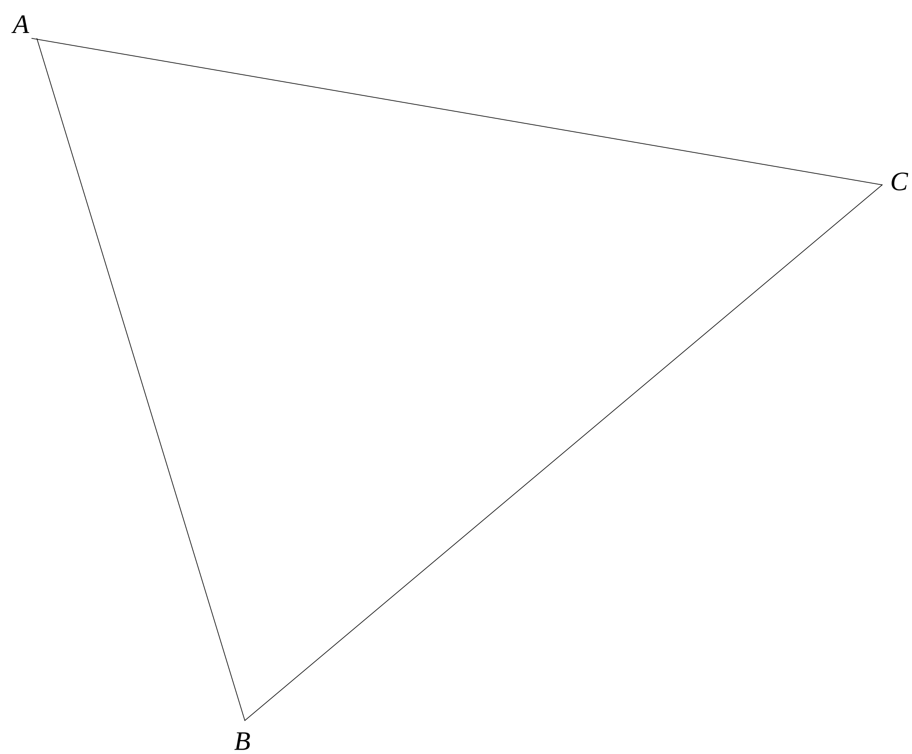
10. What other potential problems in the algorithm can you find? If you find another trouble spot, look for a way to solve the problem.

11. **Write and Reflect** Rewrite the algorithm for turning one rectangle into another. Your new algorithm should fix the three difficulties presented in Problems 7–9, as well as any others found by your classmates. Be sure to include pictures with your descriptions.

EQUAL PARTS

You can show that the two triangles are equal in area by cutting up one or both of them and rearranging the pieces to get congruent figures. Do you have another idea?

12. Divide $\triangle ABC$ into *two triangles* of equal area by making only one cut. Explain your solution.



13. Divide $\triangle ABC$ above into *four* triangles of equal area, and prepare an explanation of your solution.

You should decide how different "different" must be. For example, you might decide that making the same *kind* of dissection starting from a different place does not count as *different*.

14. Problem 13 has at least ten different solutions. By yourself or with a small group, find as many different solutions as you can.

15. Divide $\triangle ABC$ into *three* triangles that are equal in area, and prepare an explanation of your solution.

16. Using half a sheet of standard $8\frac{1}{2}'' \times 11''$ paper as your starting rectangle, cut that rectangle into:

a. two rectangles that are equal in area;

b. four rectangles that are equal in area;

c. five rectangles that are equal in area.

17. When cutting a rectangle into equal pieces, is there any number of pieces that is *not* possible? Explain your answer.

CHECKPOINT

18. For each statement below, decide whether it is true (for *all* cases!) or not. If you decide it is *not* generally true, clarify whether it is *never* true, or whether it *can* be true, but only in special cases. Justify your answer with an explanation and examples.

a. If a triangle is cut along a median, then it is divided into two triangles of equal area.

b. If a triangle is cut along an altitude, then it is divided into two triangles of equal area.

c. If a triangle is cut along an angle bisector, then it is divided into two triangles of equal area.

d. If two triangles have the same angles and the same area, then they are congruent.

e. If two triangles have the same sidelengths, then they have the same area.

f. If two triangles have the same area, then they have the same sidelengths.

g. If all three sides of one triangle are different in length from the three sides of another triangle, then the triangles will have different areas.

h. If two triangles have congruent angles, then their areas will be equal.

TAKE IT FURTHER

19. **Write and Reflect** Problem 35 in the "Take It Further" from Investigation 3.4 asked you to investigate an algorithm for dissecting any parallelogram into a rectangle. Compare that algorithm with the algorithm you've just used for turning one rectangle into another.

20. For each statement below, decide whether it is true (true for all cases) or false. Justify your answer with an explanation and examples.

- **a.** If every side of triangle B is congruent to or longer than every side on triangle A, then triangle B has the greater area.
- **b.** If every side on rectangle B is congruent to or longer than every side on rectangle A, then B has the greater area.
- **c.** It is possible to decrease the area of a triangle while increasing the length of every side.
- **d.** It is possible to increase the area of a rectangle while decreasing the length of every side.

21. Devise an algorithm that dissects one of two rectangles with the same area and rearranges it into the other.

Investigation 3.8

EQUIDECOMPOSABLE FIGURES

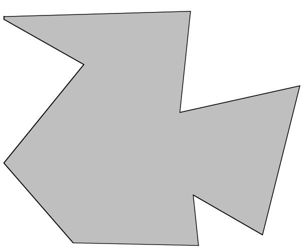
"Equidecomposable" is the traditional term for "scissors-congruent." Two shapes that have equal decompositions—that can be decomposed (cut up with scissors) into exactly the same parts—are called *equidecomposable*. Equidecomposability implies that you can *do* the cutting and rearranging, not just imagine it. That means there can only be a finite number of cuts and a finite number of pieces.

In this module, equidecomposability was first used as a kind of common-sense test for equal area. Two figures that could be dissected into each other contained the same amount of the plane. Later, you used cutting and rearranging into rectangles to devise *formulas* for calculating area, which is measured in square units.

So, if two figures are equidecomposable, they have the same area. Is the converse also true? That is, if two figures have the same area, are they necessarily equidecomposable?

1. **Write and Reflect** From your experience so far, how would you answer? Suppose you have two figures that you are *sure* have the same area. Can you cut one of them up to fit exactly on the other? Explain your reasoning.

   Consider the following cases:

   - You have a rectangle and a square with the same area. Can you cut the rectangle to fit on the square?
   - You have a circle and a triangle with the same area. Can you cut the circle to fit on the triangle?
   - You have an oddly-shaped figure with line segments for sides, like the one shown here, and you have a rectangle of the same area. Can you cut one to fit on the other?

THE BOLYAI-GERWIEN THEOREM

The following theorem may surprise you, but you now have all the tools necessary to prove it.

***Rectilinear* means that the figures must be made only of line segments; no curves are allowed.**

THEOREM 3.3 ***The Bolyai-Gerwien Theorem***

If two rectilinear figures have the same area, then they are equidecomposable.

Often, in order to prove a theorem, you need to prove several other helping theorems (called *lemmas*) first. When mathematicians *construct* a new proof, they may look first for ideas that would help their proof (if these ideas turned out to be true), and then go back later and try to prove these ideas in order to fill in the gaps in the complete proof. When mathematicians *present* a proof (either in writing or in a lecture), these helping ideas—the lemmas—are usually given first to make it easier for the audience to follow the complete proof. Below are a few important statements we will use as lemmas; you already know a couple of them. After the lemmas, you will find an outline for a proof of the rather surprising Bolyai-Gerwien Theorem.

Your job in this section is to read the lemmas and the proof outline, and to answer some questions along the way to make sure you understand the arguments.

LEMMA 1

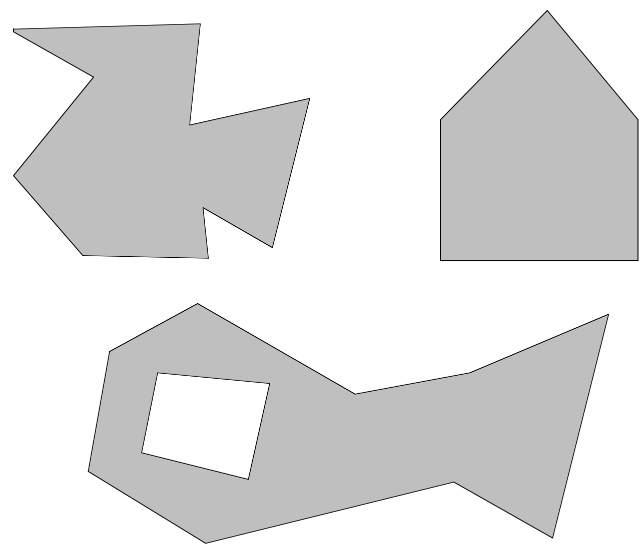
Every triangle can be dissected into a rectangle.

LEMMA 2

Every rectilinear figure can be decomposed into triangles.

You have already done a lot of work with Lemma 1; presumably you believe that it's true. What about Lemma 2?

2. Here are several rectilinear figures, including one with a "hole" in it. For each figure, find a way to dissect it into triangles. (Is there more than one possible way?)



Trace the figures and draw the "triangulation"—your scheme for cutting them into triangles—on your copies.

Even though you may already believe the lemma, the attempt to find a counterexample may lead you to a strategy for proof.

3. Lemma 2 says that *every* rectilinear figure can be decomposed into triangles. Try to invent a counterexample—some crazy rectilinear figure (perhaps one with *two* holes?) that *can't* be cut up into triangles.

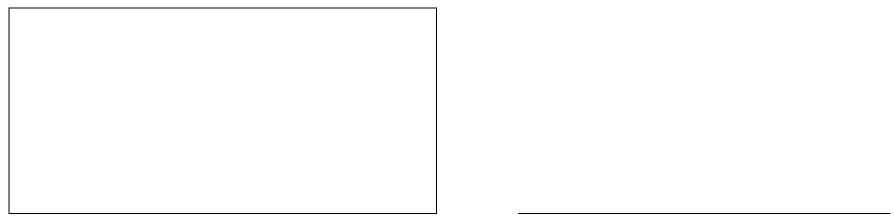
4. **Challenge** One way to prove that every rectilinear figure can be triangulated is to show how to do it. Try to find a convincing method. (A completely foolproof system is quite hard to describe.)

LEMMA 3

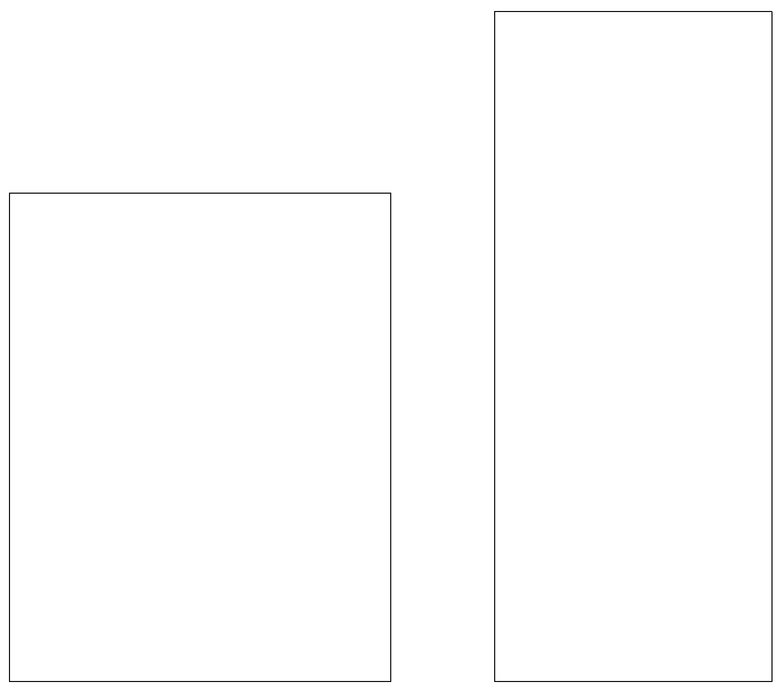
Two rectangles with the same area are equidecomposable.

You worked on this in Investigation 3.7. You may want to look back at your work from that investigation to help you with Problems 5 and 6.

5. Below are a rectangle and a line segment. Try to make a new rectangle, with the same area as the first, that has the line segment as one of its sides.



6. Below are two rectangles with the same area. Trace and cut out the figures. How can you dissect one into the other?



LEMMA 4

If figures *A* and *B* can be dissected into the same rectangle, then figure *A* can be dissected into figure *B*.

7. **Write and Reflect** Explain Lemma 4, and give clear reasons why you do or don't believe it. To do this, you might want to show two figures that can be dissected into the same rectangle, and explain how you know you could (or couldn't) dissect them into each other.

If you believe the four lemmas, then you should believe the proof below. Assume that two rectilinear figures, *A* and *B*, have the same area.

Step 1 Triangulate figures *A* and *B*.

Step 2 Dissect all the triangles from figure *A* into rectangles.

Step 3 Dissect all the triangles from figure *B* into rectangles.

Step 4 Pick a base, and dissect all the rectangles from figure *A* into new rectangles with that base.

Step 5 Use the same base from Step 4, and turn all the rectangles from figure *B* into new rectangles with that base.

Step 6 Stack all of the rectangles from figure *A* on top of each other, matching congruent bases.

Step 7 Stack all of the rectangles from figure *B* on top of each other, matching congruent bases.

Conclusion Because figures *A* and *B* had the same area, the rectangles made from them will have the same area. Because the rectangles have the same base and the same area, they must be congruent. And because figure *A* and figure *B* have been dissected into the same rectangle, the two figures are equidecomposable.

8. Draw two rectilinear figures that have the same area. Illustrate each of the steps in the proof outline above with a picture using your two figures.

9. For each of the steps in the proof, write a short justification. Refer to one of the four lemmas or write a few sentences of your own.

10. There are three different statements in the final paragraph. Explain why each of them is true.

PERSPECTIVE ON CHIH-HAN SAH

THE RESEARCH CONTINUES TODAY

Problems related to equidecomposability have been studied by mathematicians for centuries, and the interest continues today. Professor Chih-Han Sah is a mathematician working in this area. The authors asked him to tell a bit about his life and work. This is his story.

Chih-Han Sah has been a professor of mathematics at the State University of New York at Stony Brook since 1970. He was born in Beijing, China on August 16, 1934. He first visited the United States with his family in 1936 and met his eventual foster parents, William and Dorothy Everitt. In 1949, when his father died of cancer, Chih-Han Sah and his older brother came to stay with their foster parents in Urbana, Illinois. With little knowledge of English, Chih-Han was temporarily placed (on the basis of chronological age) in the 10th grade in the University High School of the University of Illinois.

With the help of teachers, fellow classmates, foster family members (and a *great deal* of hard work), he graduated in 1951 and entered the University of Illinois to study Engineering Physics. He completed the requirements for the Bachelor of Science degree in Engineering Physics and then switched to mathematics. He transferred to Princeton University in 1956 and completed his Ph.D. in mathematics in 1959.

All of these are branches of mathematics and science. Find out about one of them.

Beginning as a teaching assistant at the University of Illinois, Professor Sah also has taught at Princeton, Harvard, Pennsylvania, the University of California at Berkeley, Yale, and Columbia. His interests in mathematics have varied over the following areas:

group theory	polylogarithms
quadratic forms	combinatorial geometry
rings	algebraic problems in functional analysis
Riemann surfaces	applications to problems in electrical engineering
algebraic topology	conformal quantum field theory and
scissors congruences	statistical quantum mechanics in physics
algebraic K-theory	structures of fullerenes in chemistry

Since 1975, Professor Sah has been studying the problem of scissors congruences in higher dimensions. In particular, he has been interested in extensions to non-Euclidean spaces. Beginning in 1979, he has been collaborating with Professor Johan L. Dupont at Aarhus, Denmark, a leading expert on this problem. This is an area that has many connections with other branches of science and mathematics.

The Fermat conjecture (Fermat's Last Theorem) is a 300-year-old problem stating that there are no nonzero whole numbers x, y, and z that satisfy the equation $x^n + y^n = z^n$ when n is a whole number greater than 2. In 1641, Fermat claimed to have a proof that was too big for the margin of his book. In 1994, the conjecture was proved by Andrew Wiles, a professor at Princeton University.

One of Professor Sah's favorite accomplishments was his discovery that a problem in pure mathematics, the Fermat conjecture, is connected to a problem in statistical quantum mechanics. His discovery came about while he was working with some physics colleagues. Their computer conveniently died while working on a complicated calculation. Professor Sah took over the calculation and found an essential simplification by hand. After that, a combination of computer calculations with essential hand simplifications ultimately led to the discovery of the presence of the Fermat equations.

Since then, physicists have given some new interpretations of Fermat's Last Theorem. However, physicists are a long way from confirming their speculations by way of concrete experiments. The necessary experiments are very expensive and very delicate, so mathematics is being used to determine in which direction to go.

Investigation 3.9

AREA AND PERIMETER

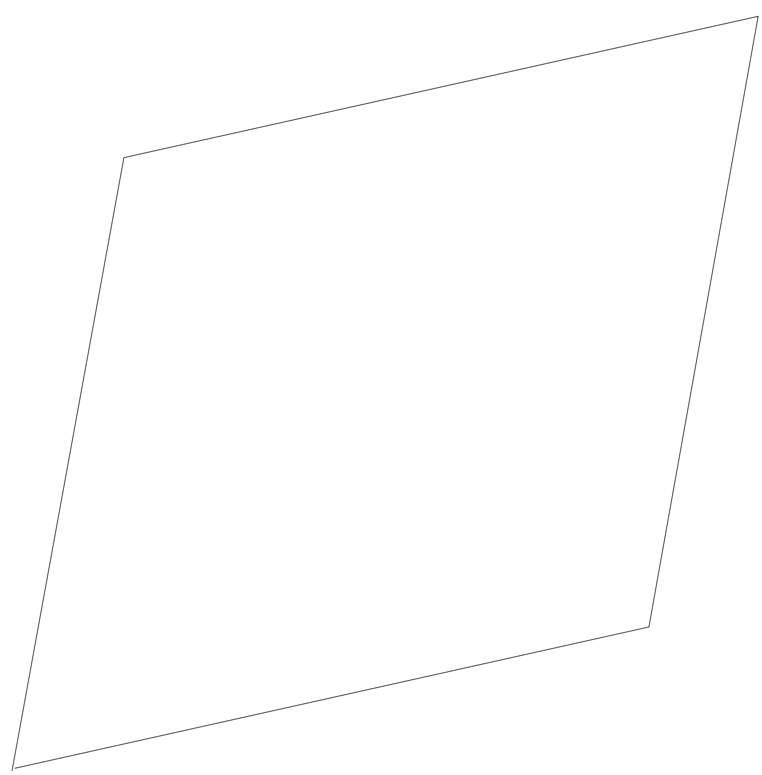
When you cut and rearrange a triangle, a parallelogram, or a trapezoid into a rectangle (or vice versa), the area is unchanged. What other properties of the figure are invariant through cutting and rearranging? You've already verified that angle measure and sidelength are not always preserved. What about the perimeter, the *sum* of all sidelengths? You probably already have some intuition about that. Now you will check that intuition.

DOES PERIMETER REMAIN INVARIANT?

1. **a.** Trace the parallelogram below or construct one of your own. Measure the lengths of its sides and calculate its perimeter.

b. Dissect your copy into a rectangle.

c. Using measurements, calculate the perimeter of the rectangle and compare it with that of the original parallelogram.



2. Compare the perimeters of the starting and ending figures when you use your algorithms for

a. dissecting a triangle into a rectangle;

b. dissecting a trapezoid into a rectangle.

3. **Write and Reflect** Summarize the results of the experiments you've just performed. Each experiment is a *specific* case, but try also to generalize from what you've seen. For each algorithm, try to answer the following questions:

a. Does *that* algorithm

i. reliably preserve the perimeter,

ii. sometimes preserve the perimeter and sometimes change it, or

iii. reliably change the perimeter?

b. If you're sure that the algorithm reliably changes the perimeter, does it

i. reliably increase the perimeter,

ii. sometimes increase the perimeter and sometimes decrease it, or

iii. reliably decrease the perimeter?

CALCULATING PERIMETER WITHOUT MEASURING

What do *qualitative* and *quantitative* mean? How are they used outside of mathematics?

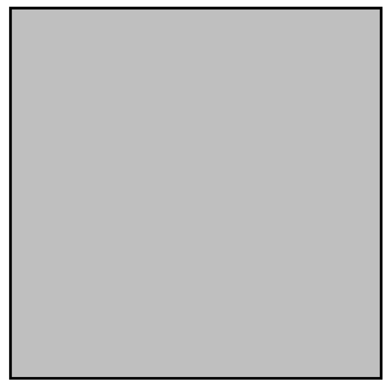
With dissections, if you know the area of the original figure, it is no effort (at all!) to find the area of the new figure. But the perimeter is not so easily determined. You now have decided, in a *qualitative* way, how perimeter is affected by each algorithm, but you have no *quantitative* way of computing a new perimeter from the old one.

Here is a dissection that changes perimeter in a regular way that allows you to compute the new perimeter from the old. In general, however, computing change in perimeter is not so straightforward.

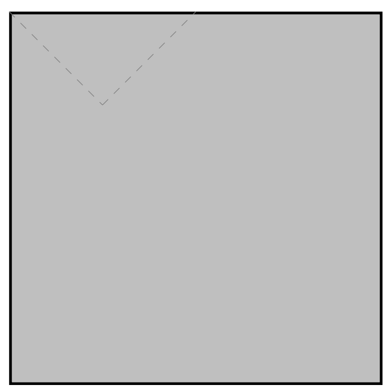
If you like to use metric measurements, 16 cm is best.

4. Construct a fairly large paper square. You'll find 8″ on a side particularly convenient.

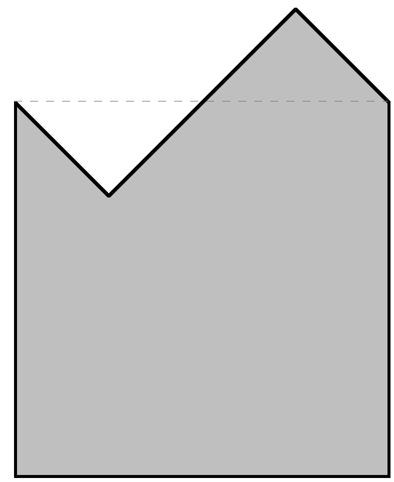
a. Record both the area and perimeter of your square; you'll need them later.



b. Construct the midpoint of the top side of your square. Using the left half of that top side as the hypotenuse, construct an isosceles right triangle.

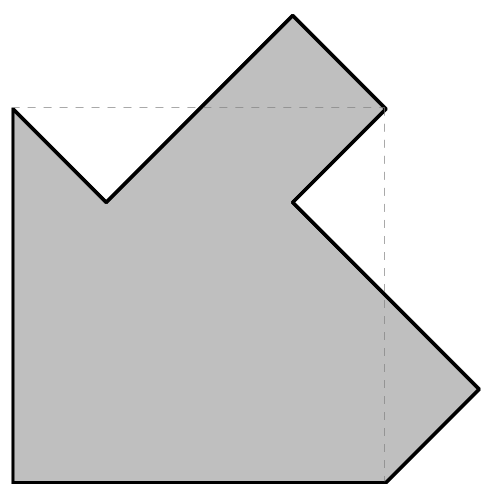


c. Cut out the triangle, rotate it around the midpoint, and tape its hypotenuse onto the remaining half of that side of the square.

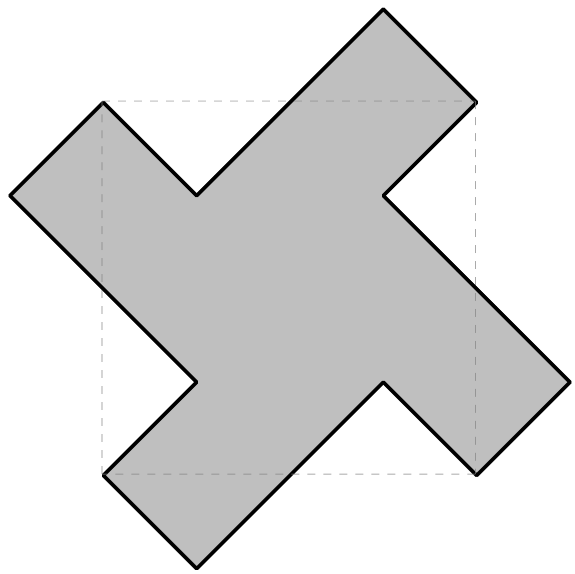


The outline of the original picture is shown as a dashed line, so it's easier to see what cuts were made and where.

d. Move clockwise to the next side of the original square and repeat the process: cut out the isosceles right triangle, rotate it, and tape it to the remaining half of that side.



e. Do the same to the remaining two sides of the original square. Your figure should now look like this:



5. What is the area of this new shape? How can you find it without further computations?

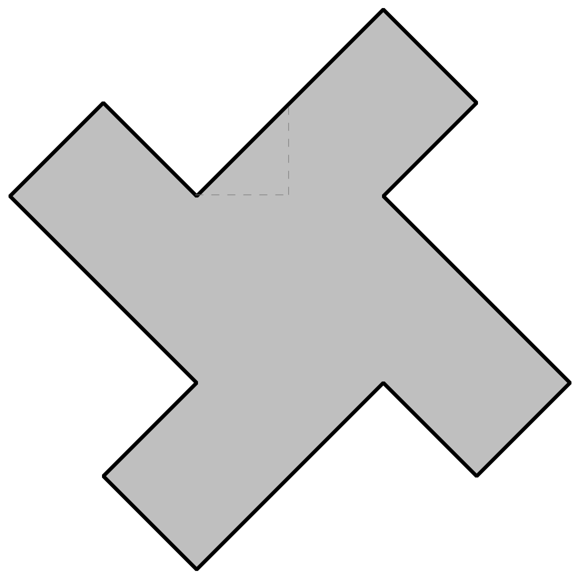
6. **a.** Is the perimeter of the new shape greater than, less than, or the same as the perimeter of the square?

b. How can you compute the perimeter of the new shape without measuring anything?

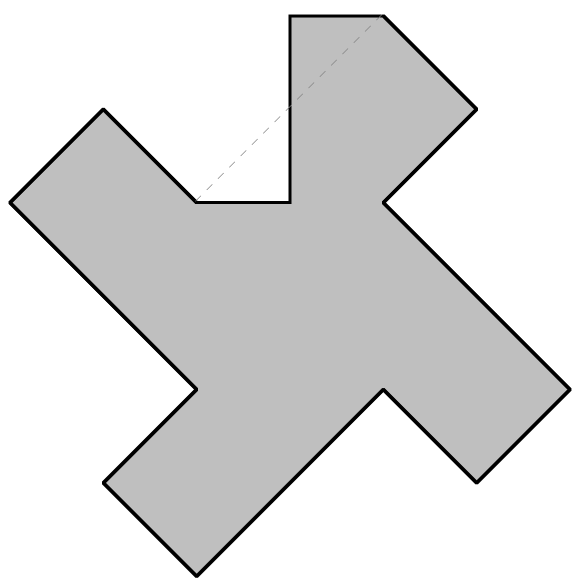
This is a difficult problem. Spend some time thinking about it. Study the construction and the picture of the final figure.

7. Repeat the process on the new figure.

a. Pick a side on the new shape, construct its midpoint, and then construct an isosceles right triangle inside the figure with its hypotenuse along the *left* half of the side you just bisected as viewed from inside the figure.

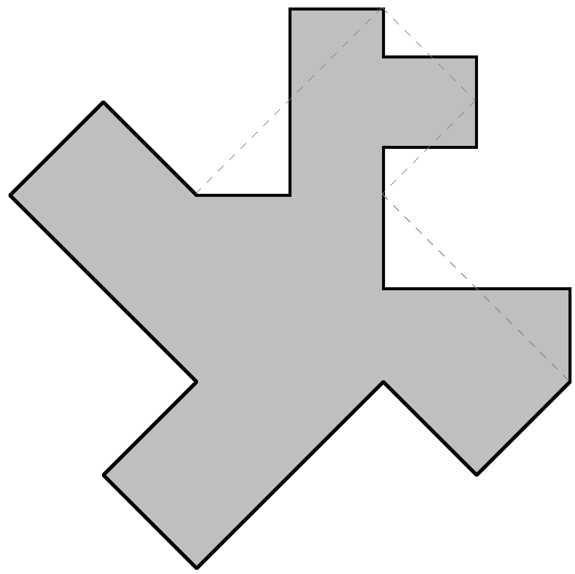


b. Cut out the triangle, rotate it, and tape it onto the other half of the side of your figure.



Make sure to move systematically (clockwise is probably easiest) around the figure, cutting the triangle out of the left-hand half of each side as viewed from inside the figure, and taping it to the right-hand half.

c. Repeat this for each side of your figure.



The dashed lines show sides that are done. See if you can complete the process.

8. What is the area of the new shape that you created? How does it compare with the area of the square you started with?

9. What is the perimeter of your new shape? How does it compare with the perimeter of the square you started with? Explain how you found the perimeter: did you measure it, use some particular process of computation, or find another way?

10. **Write and Reflect** Write a paragraph to explain what this dissection shows you about area and perimeter.

TAKE IT FURTHER. . . . .

11. Imagine continuing the process begun in Problem 4, performing it on the figure you just created, and then performing it *again* on the results of *that* dissection. How would the perimeter grow? Explain how you figured out this answer.

Investigation 3.10

MAKING THE MOST OF PERIMETER

So far, you've examined dissections in which one figure is cut up and the pieces are rearranged to fit *exactly on top of* another figure.

1. **Write and Reflect** Imagine that you can cut up a figure, rearrange its pieces, and fit them *entirely inside* another figure. What conclusions can you draw?

Drawing by hand or with a computer, you can experiment to get an answer.

2. Suppose you want to build a house with a rectangular base. The most expensive part of the project is the prefabricated outside walls you plan to use. Given your budget, you decide that you can afford 128 feet of wall. If you want to maximize the area of this 128-foot perimeter base, what shape rectangle should you use for the floor plan?

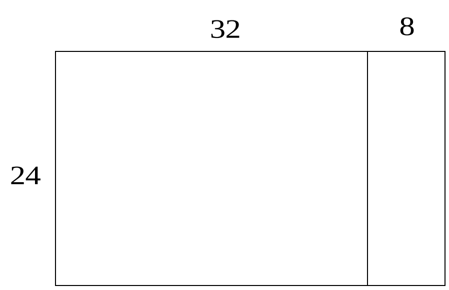
Read this proof carefully so that you can evaluate it later.

One student claimed that a 32-foot square is the solution to Problem 2, and used the following reasoning to prove it:

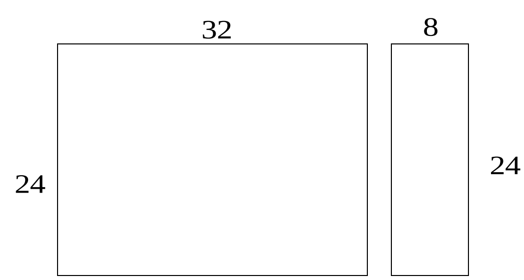
> "I'll show that a 32 × 32 square is best by demonstrating that any other rectangle with a perimeter of 128 has a smaller area than the square. I'll do this by showing that I can cut up such a rectangle and make it fit inside the 32 × 32 square with room to spare.

Is this a good example? (What's the perimeter of a 40 × 24 rectangle?)

> Suppose, for example, that I have a 40 × 24 rectangle. First, I'd cut it like this:

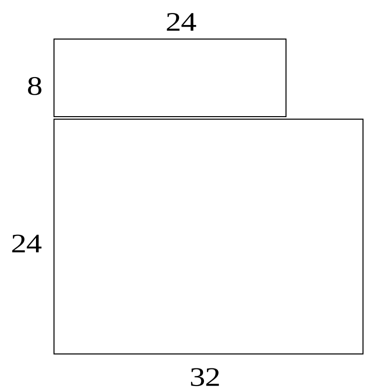


Snip!

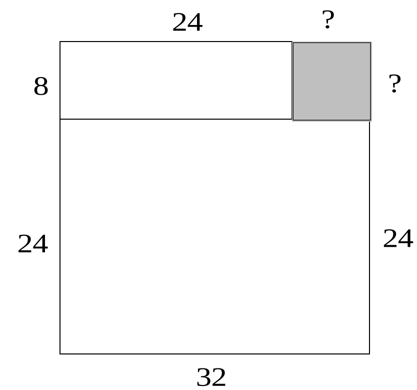


Then rearrange these . . .

"Then, I take the small strip off the side—and put it on top of the 24 × 32 rectangle:



. . . *to get this* . . .



. . . *which is not quite a square.*

"See? The operation *rearranged* the area, but I didn't add or lose any area. Meanwhile, my two pieces cover *some* of the square, but not all of it. The shaded part isn't covered, so the square has greater area! Therefore, the area of the 32 × 32 rectangle is *greater* than the area of the 40 × 24 rectangle."

These are only suggestions. The real purpose of this problem is to critique the proof.

3. Does this proof work? Can it be generalized? To help you think about what's going on, here are some questions:

a. Does the cutting argument for the 24 × 40 rectangle work? Rewrite it in your own words and explain each step.

b. Try using the same argument for another rectangle with the same perimeter of 128 feet.

c. Use the cutting argument to show that *any* nonsquare rectangle with perimeter 128 feet has a smaller area than a square with the same perimeter.

4. Generalize. Use the cutting argument to show that an $a \times b$ rectangle ($a \neq b$) has a smaller area than a square with the same perimeter.

Once you've solved Problem 4, you have a new theorem.

THEOREM 3.4

Of all rectangles with a given perimeter, the square has the greatest area.

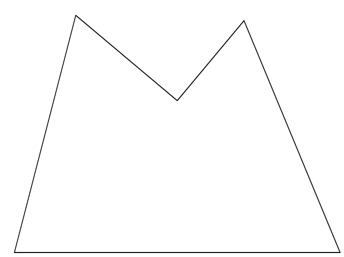
For more information about the Isoperimetric Theorem, see *Connected Geometry* module *Optimization*.

This theorem is just a small part of the *Isoperimetric Theorem*. "Isoperimetric" means "same perimeter." The Isoperimetric Theorem talks about shapes that have the same perimeter, and states which one has the greatest area.

TAKE IT FURTHER. . . . .

5. Suppose that for Problem 2 you don't care if the base of the house is a rectangle; you just want it to have four sides. Is a square *still* the best choice to maximize your area for a wall perimeter of 128 feet? Use a cutting argument to demonstrate your answer.

6. Describe a way to make the area of this polygon bigger without changing the lengths of its sides.



Geometry software may help you make a conjecture. Cutting arguments may help you develop a proof.

7. **Challenge** Of all triangles with a fixed base and height (and, as a result, the same area), which has the smallest perimeter? Find an argument to support your conclusions.

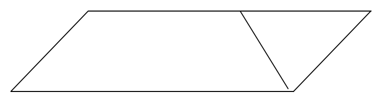
8. **Challenge** If you've done Problems 5 through 7, you're well on your way to proving this theorem: Of all polygons with a given number of sides, the regular one has the greatest area. See how far you can get toward proving this theorem.

ANALYZING DISSECTIONS

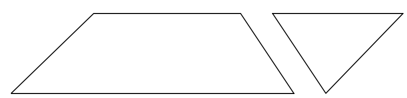
A PARALLELOGRAM CUT TO A RECTANGLE

Scene 1: *Students in Ms. Engle's geometry class have been working on ways to dissect a parallelogram and rearrange the pieces to form a rectangle of the same area. Ann explains her solution.*

Ann: I took the parallelogram and cut a triangle off one end, like this:

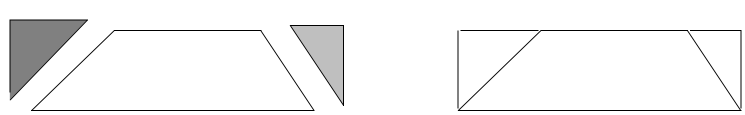


Now I have a triangle, and another shape that looks like a trapezoid.



Next I cut the triangle in two, like this:

To make a rectangle, I put one of the triangles onto one end of the trapezoid, and the other triangle onto the other end of the trapezoid.



Ms. Engle: How do you know that the rectangle has the same area as the parallelogram?
Ann: It must, since I used up all the pieces, and didn't overlap any of them.

Ann's explanation was very clear and the final figure certainly looked like a rectangle. But Ms. Engle had appointed a committee—Rafael, Michelle, and Emily—to look for real proof that the final figure is a rectangle, and to ask questions after the presentation. Answer the questions as you read the dialogue.

Rafael: How did you decide where to make the first cut? It looks like you were cutting off an isosceles triangle. Is that what you were trying to do?
Ann: I did try to make it isosceles. I thought it might not work if it were uneven. So I measured the end of the parallelogram with my ruler and then found a place to cut that had that same length.

1. Did Ann need to make the triangle isosceles? Explain.

Michelle: When you cut the triangle in two, did you cut at any special place?
Ann: I knew that I wanted to make the rectangle corners out of the pieces of the triangle, so I made my cut along an altitude.

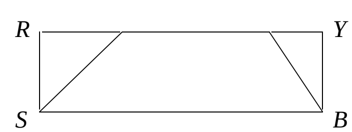
2. **a.** Define *altitude* of a triangle.

b. Explain why Ann chose to cut along an altitude.

c. How might Ann have constructed the desired altitude accurately if she didn't have a protractor?

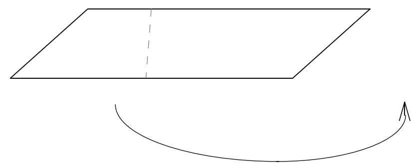
Emily: I can see why the top angles are right angles, but how do you know the bottom ones are?
Ann: Oops! I didn't think of that. I'm pretty sure they are, but I don't know why.

3. Are the lower angles ($\angle RSB$ and $\angle YBS$) in the figure below right angles? Justify your answer.



Scene 2: *Peter had worked on the same problem as Ann, but used a different method. Peter drew a sketch on the blackboard and then explained his method.*

Peter: I just made one cut, right through the parallelogram. That is the dashed line on my picture. Then I moved the left half over to the right side, and it made a rectangle.

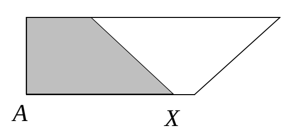


Aaron: I don't get it. Where is the rectangle? Could you just show us with the paper. The arrow doesn't make sense.

Peter: OK. Here is before . . . and here is after.

Aaron: I see it now. But I still don't think it is right. That doesn't look like a rectangle. It doesn't even have right angles.

Peter: Well, maybe I cut it a little crooked. But they really are supposed to be right angles. I know because the way I got the dashed line was by folding my parallelogram over on top of itself like this:



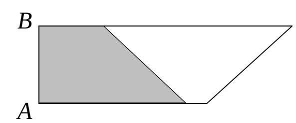
The edges line up along $\overline{AX}$, so it has to make right angles.

Aaron: That makes sense. I guess they must be right angles then.

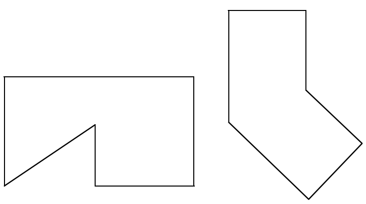
Once the class understood what Peter had done, the debate centered on whether the final figure was actually a rectangle. Here are some of the issues in that debate. All of these questions must be resolved if you are to be certain the last figure is a rectangle.

Read each one carefully, and answer it in writing. After you have written your answers, trade with another student, and critique your partner's answers thoughtfully.

4. Peter folded the bottom edges carefully to make a right angle at *A*. How do you know that this also makes right angles along the top edge at *B*?

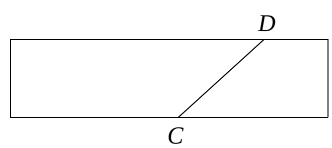


Here are two figures that each have four right angles.



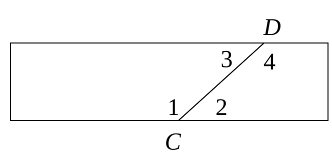
5. Once Peter establishes that his figure has four right angles, has he proved that it is a rectangle?

6. Peter joined the two pieces at $\overline{CD}$. What proof do you have that the two edges joined at $\overline{CD}$ are the same length?



7. What might Peter's rectangle look like if the edges he joined were different lengths? Make a sketch.

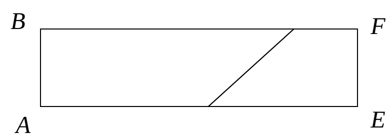
8. When Peter put the two pieces together at $\overline{CD}$, he assumed they made a straight line along the top and the bottom of the "rectangle." Prove that joining angles 1 and 2 makes a straight line, and that joining angles 3 and 4 makes a straight line.



9. In a rectangle, opposite sides have the same length. Explain how you know that this is true in Peter's "rectangle."

AB = *EF* because:

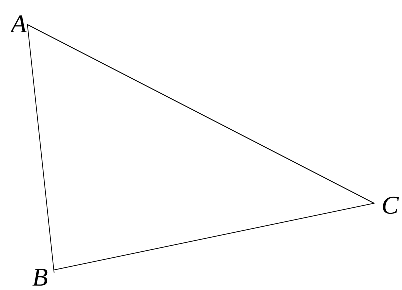
AE = *BF* because:



A Problem and a Proposed Solution

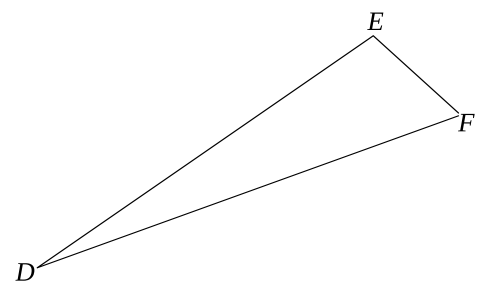
Jeremy was presented with this problem:

> Construct a triangle, say $\triangle DEF$, that is equal in area to $\triangle ABC$ below, but that has no angle congruent to any angle of $\triangle ABC$.



In his response, he said:

> I measured the sides of the triangle. They were 5 cm, 4.2 cm, and 3.3 cm, which added up to 12.5 cm. So I made a new triangle that had sides 1.5 cm, 5.25 cm, and 5.75 cm. All the angles are different, but the sides add up the same.

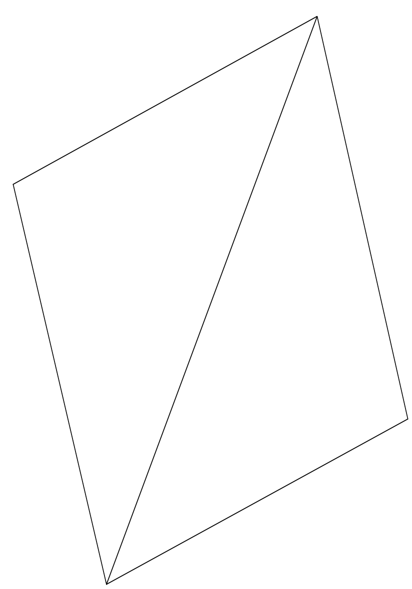


Jeremy's triangle

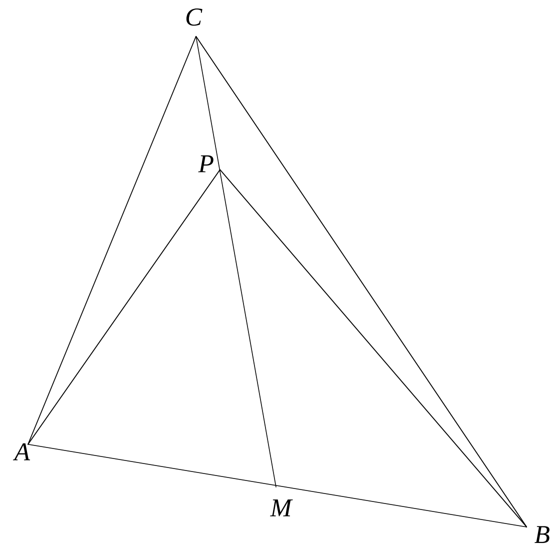
10. **Write and Reflect** Do you believe that Jeremy's construction solves the problem? Write either a defense or a critique of his approach.

SUPPLEMENTARY PROBLEMS

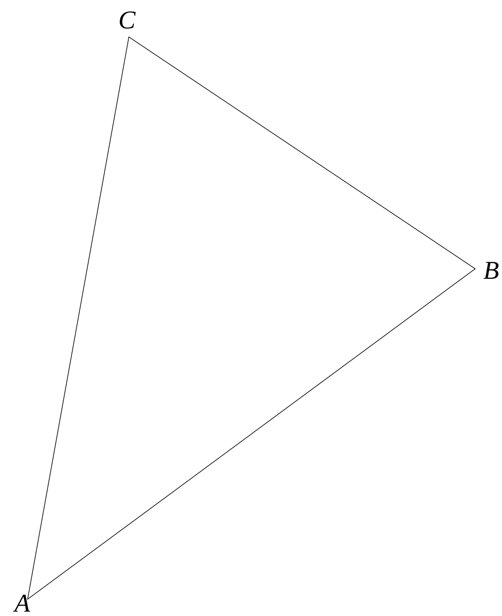
11. Show that a parallelogram can be dissected and rearranged to form a rectangle having one base congruent to a diagonal of the parallelogram.



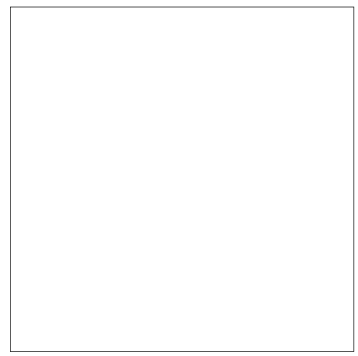
12. $\overline{CM}$ is a median. P is a point on $\overline{CM}$. Show that $\triangle APC$ is scissors-congruent to $\triangle PBC$.



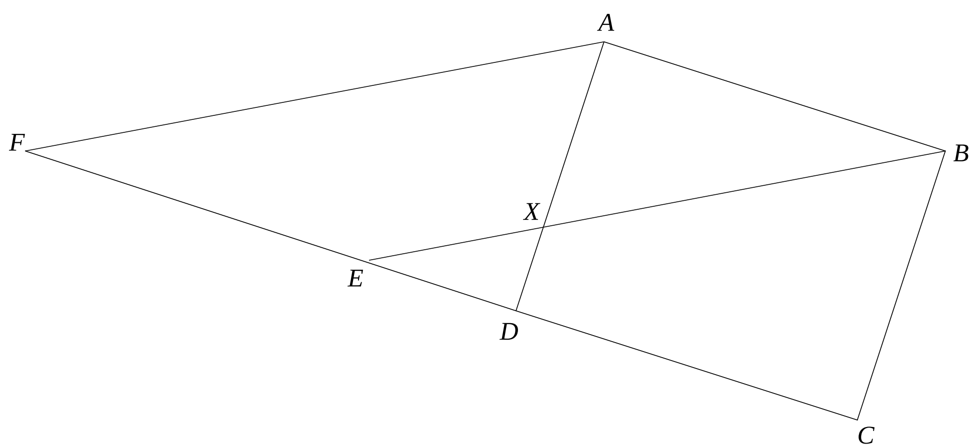
13. Cut $\triangle ABC$ into a rectangle having a height different from any of the three heights (altitudes) of the triangle.



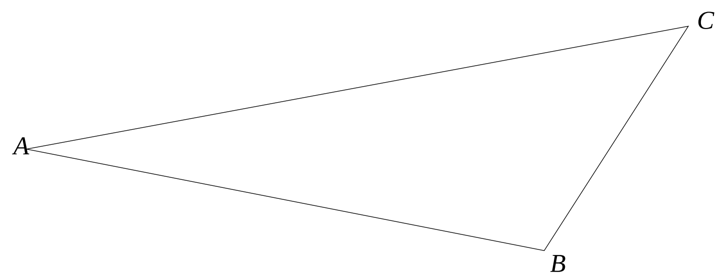
14. Cut a square into a rectangle that has a base congruent to one of the square's diagonals.



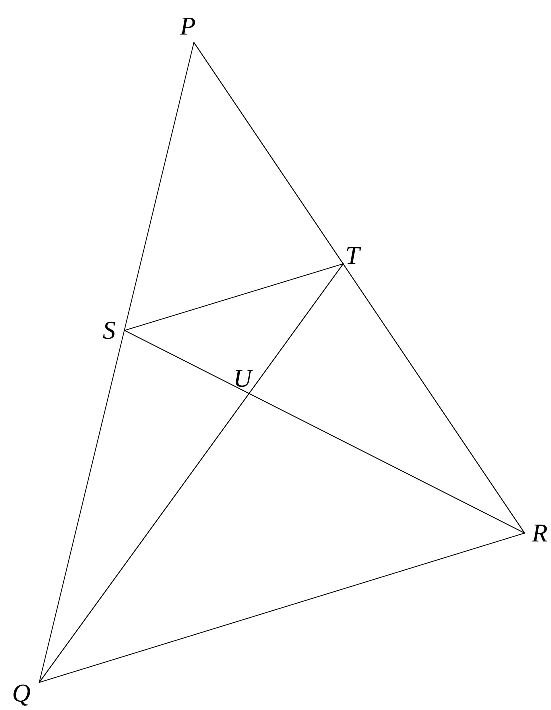
15. *ABCD* is a rectangle; *ABEF* is a parallelogram; C, D, E, and F are collinear; $\overline{AD}$ and $\overline{BE}$ intersect at X. Show that *BXDC* is scissors-congruent to *AXEF*.



16. Make a cutting argument to demonstrate that $\triangle ABC$ is scissors-congruent to some right triangle.

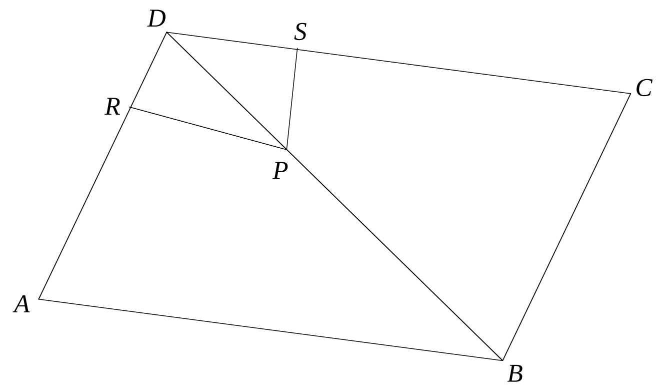


17. S and T are points on $\overline{PQ}$ and $\overline{PR}$ of $\triangle PQR$. $\overline{ST}$ is parallel to $\overline{QR}$. $\overline{SR}$ and $\overline{QT}$ intersect at U. Show that $\triangle SQU$ is scissors-congruent to $\triangle TRU$.

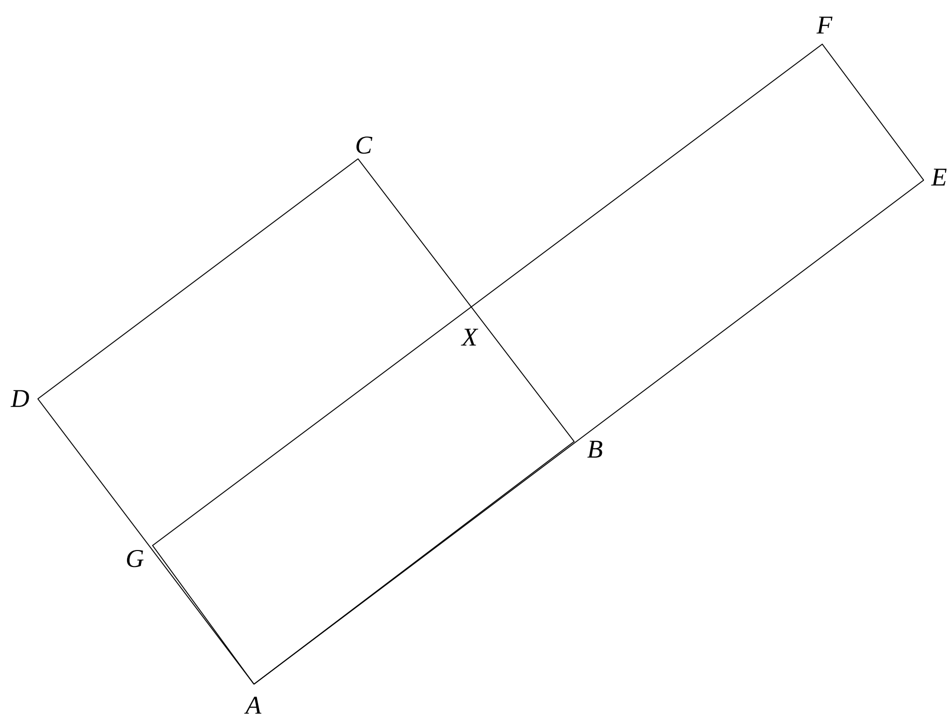


Two points are *equidistant* from a line if the perpendiculars drawn from the points to the line are equal in length.

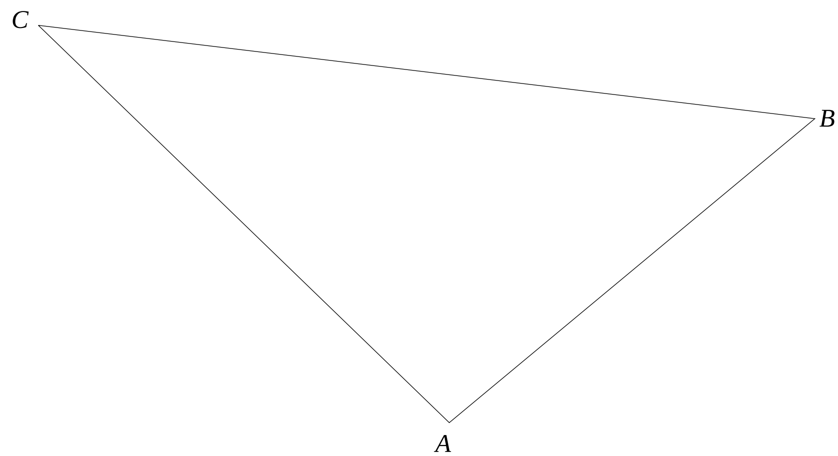
18. *ABCD* is a parallelogram with diagonal $\overline{BD}$. *P*, *S*, and *R* are points on $\overline{BD}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that *S* and *R* are equidistant from $\overline{BD}$. Show that *BARP* is scissors-congruent to *BCSP*.



19. Rectangles *ABCD* and *AEFG* are scissors-congruent. $\overline{BC}$ and $\overline{FG}$ intersect at *X*. What conclusions can you draw and why?



20. Make two (and only two) cuts on $\triangle ABC$ and rearrange the pieces to form a rectangle having base $\overline{AB}$.

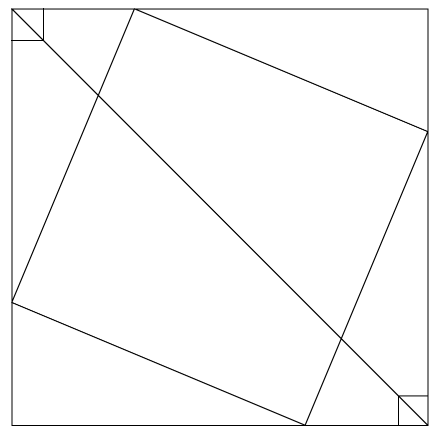


21. On the next page is a "picture proof," a proof without words. Here is the statement of the theorem it claims to prove:

THEOREM 3.5

The angle bisector of the right angle of a right triangle divides the square constructed on its hypotenuse into two equal-area quadrilaterals.

Write an explanation, using facts you know about dissection, about why this picture is a convincing argument for this theorem. In other words, complete the proof by adding a written argument.



Investigation 3.12

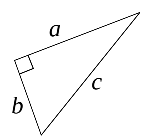
PYTHAGOREAN CUTTING PROOFS

In Investigation 3.6, you studied the Pythagorean Theorem:

THEOREM 3.2 ***The Pythagorean Theorem***

In a right triangle, the square built on the hypotenuse is equal to the squares built on the other two sides.

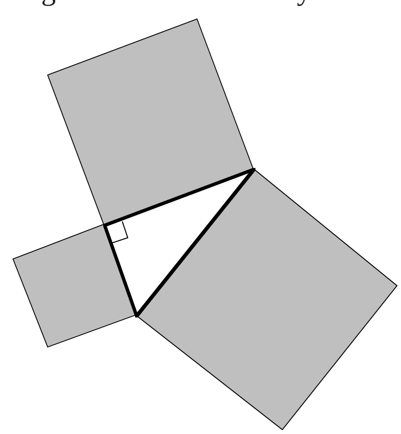
You've also seen it expressed this way:



$$a^2 + b^2 = c^2.$$

The Pythagorean Theorem is a major result in mathematics. You saw one proof of the theorem in Investigation 3.6. In this investigation, you will look at several more proofs.

1. Here is a picture that shows squares on each side of a right triangle. Explain in writing what the Pythagorean Theorem tells you about the squares.



Why do people bother finding new proofs for old results?

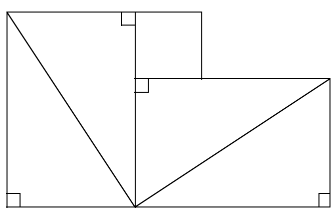
Throughout the centuries, thousands of proofs of the Pythagorean Theorem have been found, some of which don't use any words at all. This investigation contains five proofs of the theorem, all of them proofs by picture (sometimes with a little algebra thrown in).

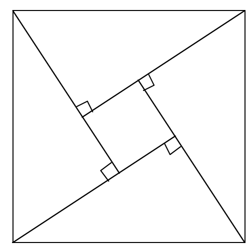
Your job is to select one of the proofs, work through it so that you understand *why* it proves the Pythagorean Theorem, and then explain the proof in a class presentation.

PROOF 1: BHĀSKARA

Pythagorean triples are integer lengths that will build a right triangle. So Bhāskara looked for sets of three integers (a,b,c) where $a^2 + b^2 = c^2$. The numbers (3,4,5) are a well-known example of a Pythagorean triple. It is less well-known that *every* integer is part of some Pythagorean triple. Can you find any integers a, b, and c so that $a^3 + b^3 = c^3$?

Bhāskara Acharya (1114–1185) was a Hindu mathematician. He published a book called *Lilavati*, named after his daughter. Bhāskara's work was primarily in algebra, for example, solving quadratic equations and looking for Pythagorean triples (see Problems 28–32 in Investigation 3.6). Below is his famous one-word proof of the Pythagorean Theorem.



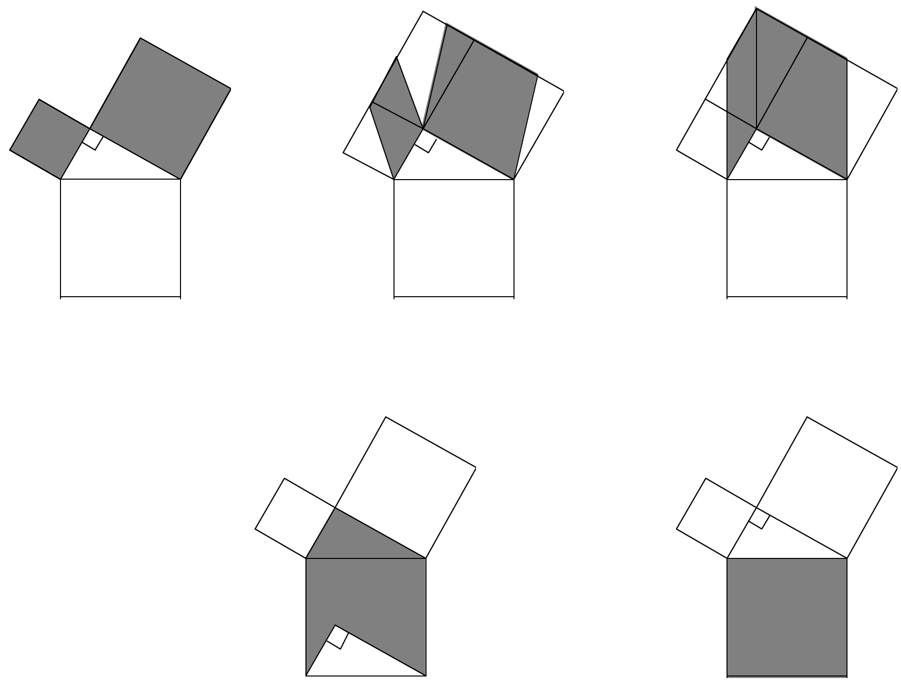


Behold!

PROOF 2: BASED ON EUCLID'S PROOF

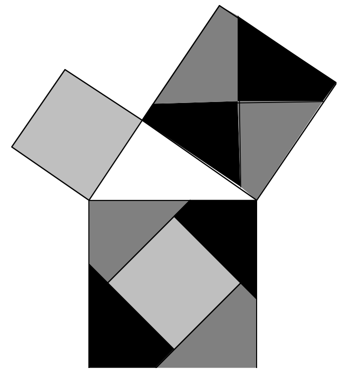
The following dissection proof is based on Euclid's proof of the Pythagorean Theorem, presented as Proposition 47 in Book 1 of *The Elements*, Euclid's classic geometry texts.

The Elements (13 books in all) are the best-selling mathematics books of all time.



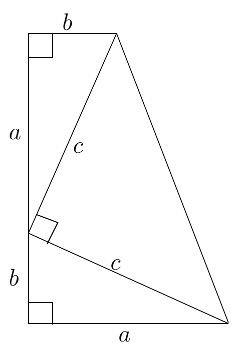
PROOF 3: HENRY PERIGAL

Henry Perigal was a stockbroker who lived in London in the 19th century. He discovered the following dissection proof of the Pythagorean Theorem around 1830, and he liked it so much that he had the diagram printed on his business cards. Perigal was also an amateur astronomer; his obituary in the 1899 notices of the Royal Astronomical Society of London described one of this pet peeves: He wanted to convince people that it was terribly incorrect to say the moon "rotates" rather than "revolves" around the Earth.



PROOF 4: JAMES GARFIELD

James Garfield (1831–1881) was the 20th President of the United States. Five years before becoming President, he discovered the following proof of the Pythagorean Theorem.

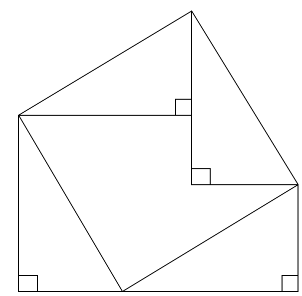


$$
\begin{aligned}
\tfrac{1}{2}(a+b)(a+b) &= \tfrac{1}{2}(ab) + \tfrac{1}{2}(ab) + \tfrac{1}{2}c^2 \\
a^2 + 2ab + b^2 &= 2ab + c^2 \\
a^2 + b^2 &= c^2
\end{aligned}
$$

TAKE IT FURTHER

2. **A Rhyming Reason** Here is another, more difficult "picture-proof" of the Pythagorean Theorem. This was probably devised by George Biddel Airy (1801–1892), an astronomer. The poem that accompanies the picture appears beside it. See if you can reason through this proof and poem.

Here I am as you may see,
$a^2 + b^2 - ab$.
When two triangles on me stand,
Square of hypotenuse is planned.
But if I stand on them instead,
The squares of both the sides are read.



3. **What if They Aren't Squares?** The Pythagorean Theorem makes a statement about the areas of squares built on the sides of a right triangle (namely that the two smaller areas sum to the third). What if the shapes built aren't squares? Suppose, instead, you constructed semicircles on the sides of a right triangle? Or equilateral triangles? Or rectangles? Construct various shapes on the sides of a right triangle, and explore the cases when it is possible to relate the three areas in some way.

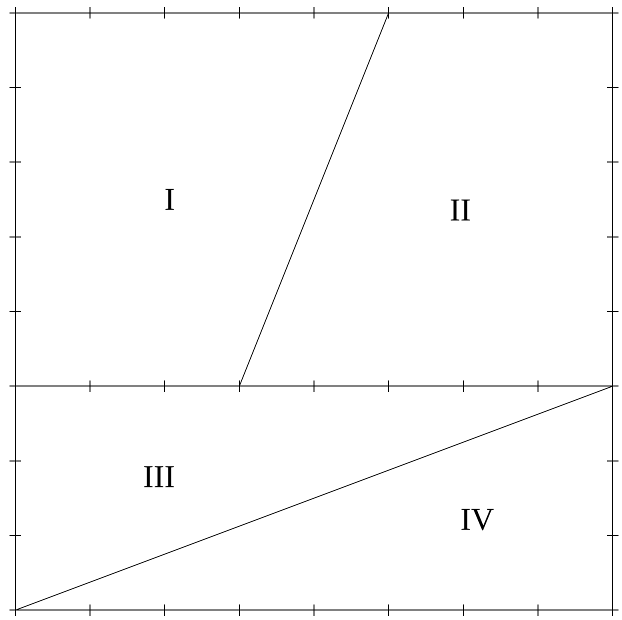
Investigation 3.13

A CUTTING PARADOX

Lewis Carroll, author of *Alice in Wonderland* and *Alice through the Looking Glass,* was also a mathematician with a strong interest in logic and logical puzzles. The puzzle shown here is said to have been one of his favorites.

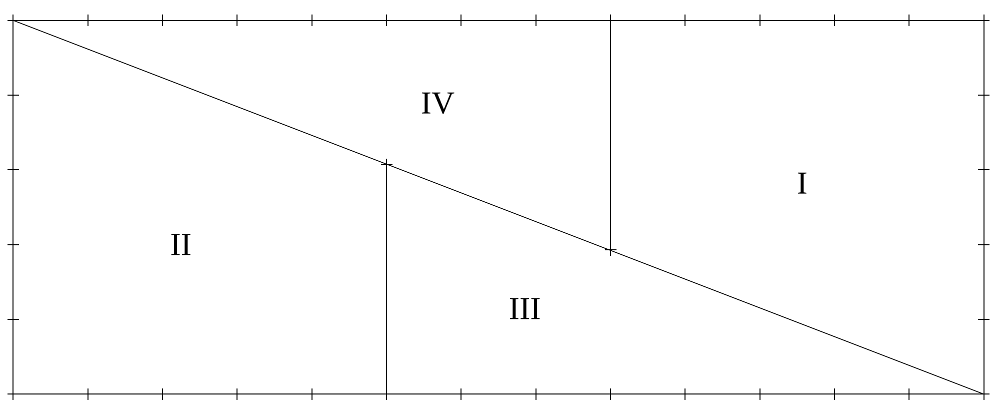
You have been working under the assumption that dissection-and-rearrangement preserves area. Here is a dissection that seems to be a counterexample: area seems to *change* when a shape is cut up and rearranged. Could it be? What's going on?

1. *Carefully* copy the square below (you may want to use graph paper). It is eight units on a side. *Carefully* cut your copy into four pieces as indicated by the sketch:



What is the area of an 8×8 *square?*

2. Now rearrange the pieces into the following figure:



What is the area of a 5×13 *rectangle?*

Your earlier proofs were all to show that things were what they seemed to be. Here is a place where proof is essential, but in an upside-down kind of way. You *don't* want to believe the result, so you want to show that something is *not* what it seems to be.

What does *paradox* mean, and why is the cutting problem you just saw not really a paradox?

3. **Write and Reflect** The same four pieces seem to make either a square or a rectangle. Yet, if you compute the two areas by formula, the areas are not the same! Figure out what has gone wrong, and explain how it happened.

This "paradox" shows why it's important to reason about cutting and rearranging, rather than just making things look right. Sometimes your eyes can deceive you, and you can end up with unexpected—and incorrect—results.

The trick works because the numbers used for the square and rectangle have a special property: $8 \times 8 = 64$ and $5 \times 13 = 65$. The area difference is just one square unit. These numbers (5, 8, and 13) belong to a special sequence called the Fibonacci sequence. You may have seen this sequence before, and you will almost certainly encounter it again. Fibonacci numbers appear in a surprising number of places, both in mathematics and in nature.

Here is the beginning of the Fibonacci number sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

4. Examine the sequence. Figure out a rule that generates the next number in the list. List the next three Fibonacci numbers.

5. Write out the rule to generate Fibonacci numbers, either in words or in symbols.

These numbers have many special properties. The one that makes the cutting "paradox" so convincing is this: if you take any three consecutive Fibonacci numbers (like 5, 8, and 13) and square the middle one, the result will be just one more or one less than the product of the two outer ones. In the example you've already seen, 8^2 is just one less that 5×13. You can use this property to build different-sized "cutting paradoxes."

6. Using graph paper or measuring carefully, create a 5×5 square. Can you find a way to make it turn into a 3×8 "rectangle"? (The cuts will look similar to those you did before.)

7. Pick a larger Fibonacci number, and draw a square with that number of units on each side. Then turn it into a "rectangle" whose sides are equal to the Fibonacci numbers on either side of the original number you picked.

PERSPECTIVE ON FIBONACCI

This essay will introduce you to a famous Italian mathematician, known to us as Fibonacci, and his famous rabbit problem, which you will solve.

Leonardo de Pisa, also known as Fibonacci, lived from about 1175 to 1258. He was the first European to use the Hindu-Arabic numerals 0–9, the numerals we use today. As a young boy, Fibonacci traveled with his father. He may have learned of the Hindu-Arab numbering system on visits to North Africa. He noticed how easy calculations were, compared with the Roman numerals he had been using, so he wrote a book called *Liber Abaci* about calculating with this new number system.

In *Liber Abaci*, Fibonacci introduced what was to become a famous problem in mathematics:

Suppose you have two baby rabbits in a cage, one male and one female. Suppose that rabbits must be one month old to reproduce, and that after that they will have a pair of babies (again one male and one female) at the end of every month. How many pairs of rabbits will you have after a year has gone by, assuming none of them die?

You probably don't know a solution to this problem just from reading it, but if you work through the problem you may notice something surprising.

In each row, pairs that are old enough to reproduce are in capital letters; pairs that aren't old enough are in lower case. Each pair's babies (a new pair of rabbits) is shown on the line with the parents. The next month, when they are old enough to reproduce, they move onto their own line.

8. Work through Fibonacci's rabbit problem. A table like the one below might help. How many pairs of rabbits will you have after 12 months?

Month	Rabbits	Total Pairs
Start	pair 1	1
1 Month	PAIR 1	1
2 Months	PAIR 1—pair 2	2
3 Months	PAIR 1—pair 3 PAIR 2	3
⋮		

9. Describe the pattern in the number of pairs of rabbits. Why does *that* pattern occur in *this* problem?

TAKE IT FURTHER.................................

10. Fibonacci numbers "appear" in pinecones, pineapples, sunflowers, and other natural objects. Find out how they appear and prepare a poster or find some items that show how the Fibonacci numbers appear in nature. Be prepared to explain where the Fibonacci numbers are hidden in these objects.

11. If you take the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, . . . and make a new sequence by subtracting any number from the one after it, you get the following pattern:

$$\begin{aligned} 1 - 1 &= 0 \\ 2 - 1 &= 1 \\ 3 - 2 &= 1 \\ 5 - 3 &= 2 \\ 8 - 5 &= 3 \\ 13 - 8 &= 5 \\ 21 - 13 &= 8 \\ 34 - 21 &= 13 \\ &\vdots \end{aligned}$$

12. Instead of subtracting each term from the next, what if you take the *ratios* of consecutive terms:

$$\begin{aligned}\frac{1}{1} &= 1\\ \frac{2}{1} &= 2\\ \frac{3}{2} &= 1.5\\ \frac{5}{3} &\approx 1.667\\ \frac{8}{5} &= 1.6\\ \frac{13}{8} &= ?\\ \frac{21}{13} &= ?\\ \frac{34}{21} &= ?\\ &\vdots\end{aligned}$$

Use a calculator or computer to fill in a few more ratios. The ratios seem to be approaching a fixed value. How closely can you approximate this number?

PERSPECTIVE ON THE FIBONACCI SEQUENCE

This essay will help you learn more about the famous Fibonacci sequence. You will see that it is much easier to define the Fibonacci numbers that form this sequence recursively than with an explicit formula.

In words, the Fibonacci sequence is often described this way: "Start with 1, 1. Then get the next number by adding the two previous numbers." In symbols, you can write it this way:

$$\begin{aligned} f_1 &= f_2 = 1\\ f_n &= f_{n-1} + f_{n-2}.\end{aligned}$$

(f_1 means the first Fibonacci number, f_2 means the second Fibonacci number, and f_n means the Fibonacci number in the nth position in the sequence.)

These symbols give a "recursive definition" of the Fibonacci numbers: they tell you how to find the next *if* you know the earlier ones. But if you want to know, say, the

100th Fibonacci number, you'd need to figure out the 99th and 98th; to know them, you'd need to know the 97th and 96th . . . . It would be handy to have a simple formula into which you could substitute 100, and out would pop the 100th Fibonacci number.

It's amazing that a sequence so easy to describe is so hard to write a formula for. It took nearly 500 years after the death of Leonardo de Pisa before a formula for his sequence was finally discovered.

The formula to find any Fibonacci number looks like this:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Before powerful computers were invented, direct formulas were usually preferred over recursive definitions because the computation was simpler: you can just substitute 100 into the formula, rather than working out the first 100 terms. In this case, however, the formula is complicated, difficult to find (it did take 500 years!), and difficult to understand.

Today, recursive definitions are becoming more and more important. As in this example, they are often much easier to understand than formulas. And with computers doing the tedious work, ease of computation is no longer as much of an incentive as it used to be for finding other methods.

Investigation 3.14

Cutting Up Solids

All of the objects you touch are three-dimensional, so it is useful to ask geometric questions about these objects. What cross sections (flat slices) can you make through three-dimensional objects? Can you cut up three-dimensional objects and rearrange the pieces to form other familiar three-dimensional objects?

You will need clay or some other material to build models of three-dimensional shapes and a way to slice the models (dental floss works well with clay).

Cross Sections

Cross sections of three-dimensional objects are the two-dimensional shapes you see when you perform a single, flat slice through the object. Visualizing "slices," or cross sections of three-dimensional objects can be difficult, even for experienced geometers. In this activity, you will actually *make* the slices.

For each problem, try to picture the answer (and write it down) before slicing with your materials. It is likely that your intuition and visualization skills will improve as you work through the problems.

1. **A Sphere** What cross sections can you get from a sphere? Make a ball from whatever material you are using, and slice it. What shape is the cross section? Try different-sized balls and different slices.

- **a.** Describe what shapes you get. How much variety can you get with one ball?
- **b.** Imagine that someone says "Perhaps if you sliced at a different angle, you'd get a different shape from those you've seen." Present an argument to explain why this could not be true for a sphere.

2. **A Cylinder** Picture a cylinder.

- **a.** How can you slice through it to get a rectangular cross section? (Describe the slice in words, demonstrate it, or draw a picture of it.)
- **b.** What other shapes can you get by slicing a cylinder?

If you use potatoes or sponges for these slices, you can use the cross sections as ink stamps.

3. **A Cube** What slices can you get from a cube? Make a cube from your material. For each of the following shapes, try to *picture* how it might be made. Then try

to make it by slicing a cube. Try to explain your result. When a shape can be made, say how; when you think some shape can *not* be made, say why.

a. A square

b. A rectangle that is not a square

c. A parallelogram that is not a rectangle

d. A trapezoid

e. An equilateral triangle

f. A triangle that is not equilateral

g. A pentagon

h. A hexagon

i. An octagon

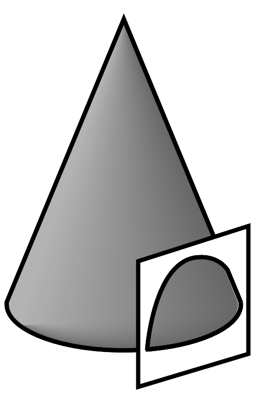
j. A circle

k. Can you make any other shape not listed above?

4. **Cube 2** You *can* get quite a variety of triangles by slicing a corner off a cube, but not *all* triangles. Describe the types of triangles that can *not* be made in this way.

5. **A Tetrahedron** Use your material to make a regular tetrahedron. Can you make a slice through the tetrahedron so that the cross section is a square? What other shapes can you make by slicing the tetrahedron?

A cone has a circular base and a point at the vertex. Think of an ice cream cone.



6. **A Cone** At the side is a picture of a cone. What cross section can you make with it? Draw the slices you can make, and name them if you can. (These shapes, called *conic sections*, are very famous in mathematics. You will undoubtedly see them again.)

7. **A Doughnut** It is quite a challenge to picture some of the possibilities. Try some experiments. If you use real doughnuts, try to discount the irregularities in their shapes (and don't eat too much!).

EQUIDECOMPOSABILITY

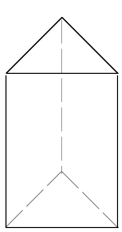
Your experience cutting up solid figures and looking at the cross sections may give you some useful intuitions about dissecting solid figures and rearranging the parts to make new solid figures, similar to what you've done with flat (two-dimensional or "plane") figures.

FOR DISCUSSION

When you dissected plane figures and rearranged the pieces, perimeter and shape often changed, but area did not. What measurements do you expect will change when you dissect a solid figure and rearrange the pieces? What might be some invariants?

It's tempting, when working with clay, to squish things together to make them fit. No fair! The goal is to make the faces fit together simply by the way you make the cuts.

A *prism* is a geometric solid with two opposite faces (the bases) that are congruent polygons.



A triangular prism

8. Make a cube with your materials. Find some way to cut the cube and rearrange the pieces to make a rectangular solid that is not a cube.

9. What other geometric solids can you make by cutting a cube and rearranging the pieces? Make at least two more figures.

10. Make a prism that has parallelograms as its bases. Can you cut this prism and rearrange the pieces into a rectangular solid? Explain.

11. Make a prism that has a triangle as its base. Can you cut this prism and rearrange the pieces into a rectangular solid? Explain.

12. Make a regular tetrahedron with your materials. Can you cut this tetrahedron and rearrange the pieces into a rectangular solid? Is this problem easier or harder than Problem 11? Why?

PERSPECTIVE ON THE HILBERT PROBLEMS

Mathematicians can make an important contribution by posing interesting problems, as well as by solving them. This essay introduces some famous problems posed by David Hilbert that led to a great deal of important mathematical research.

No one can predict which mathematical and scientific questions will have the most impact on people's lives in the future, but scientists need some way to decide what to research. They research the questions that are interesting and that have the potential for being important.

In 1900, an international congress of mathematicians was held in Paris. David Hilbert, one of the most famous mathematicians of his time, gave a speech at the congress entitled "Mathematical Problems." In it, he planned to present 23 questions that he thought would be among the most interesting and most fruitful for mathematicians to research in the coming century. This collection of problems is called the "Hilbert Problems," and the individual problems are often referred to by number, as in "Hilbert's third problem." Many of the problems have indeed been solved in the 20th century, and a rather large book has been published called *Mathematical Developments Arising from Hilbert Problems*.

Hilbert's third problem was called "The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes." Professor Hilbert knew the Bolyai-Gerwien Theorem (see Investigation 3.8), but the question about equidecomposable figures in three dimensions was unanswered. If two solid figures have the same volume, can you cut up one of them and rearrange the pieces to form the second? Hilbert suspected that the answer, in general, was "no," so he offered this challenge to mathematicians: "Find two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra."

The third problem was never actually presented at the Paris congress, however. As the meetings got underway, a heat wave descended on the city. Professor Hilbert had to cut his speech short, so he presented only ten of his 23 problems. He intended to publish the complete compilation of problems, but by the time it reached publication, Max Dehn, one of Hilbert's students, had solved the third problem.

13. Write and Reflect What if someone *did* meet Hilbert's challenge and find two such tetrahedra? Why would that be sufficient proof that solid figures with the same volume are not necessarily equidecomposable?

Max Dehn didn't solve the problem exactly as Hilbert had stated it; instead he proved that, "A cube and a regular tetrahedron having the same volume are not equidecomposable." No wonder our Problem 12 was so hard!

Investigation 3.15

CUT A RUG AND OTHER DISSECTIONS

WITCHES, INC.

How do you cut a piece of cloth to make a cone-shaped witch's hat? One class created a fictitious company called "Witches, Inc.," whose task it was to produce witch's hats. Can you figure out how to make them?

To fit your head, the circle at the hat's base should be the same circumference as your head. Maybe you'll want to make a brim, too.

1. First, try to make a witch's hat to fit your own head. Be prepared to present your solution to the class. You will probably want to use something like construction paper instead of cloth.

2. Now that you have figured out how to make a witch's hat for yourself, find a partner. The two of you can try to make hats for each other. What are the differences between the two hats? What changes did you have to consider when you made the hat for your partner?

3. People's heads vary in size. Try making a witch's hat to fit a circumference of 20 inches. How about a circumference of 24 inches?

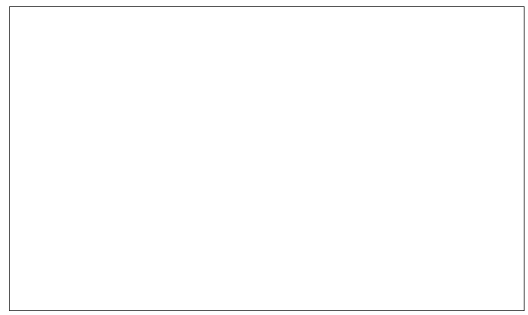
4. **Write and Reflect** As a hatmaker, you don't want to do new calculations each time you get a customer with a different head size. You need a general rule to tell you how to make a witch's hat if you know the head size. You may need to make a few more hats before you can come up with a rule. Once you think you could make a hat for any size head, write up your rule (be sure to use pictures!) to share with the rest of the class.

BIRTHDAY CAKE

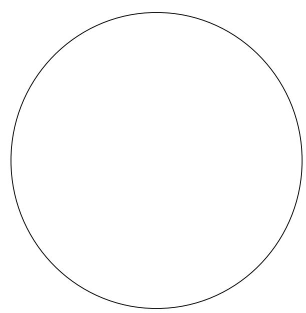
Cutting birthday cake is tricky. You have to be sure that everyone gets the same size piece. Try some of these cake-cutting problems.

5. How do you cut this cake into four pieces with the same amount of cake in each? How many different ways can you find? (Cutting one way may give different

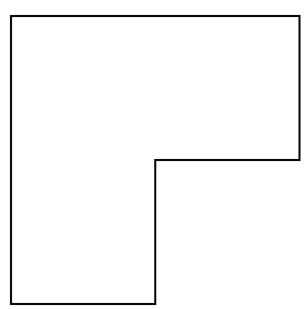
shapes from cutting another way, without changing the sizes of the pieces. Why is that?) What about the same cake into three pieces of equal amounts?



6. How do you cut this cake into ten same-size pieces? What is the fewest number of cuts you can do it in?



7. How do you cut this cake into three same-size pieces? Into four pieces? Into twelve pieces?

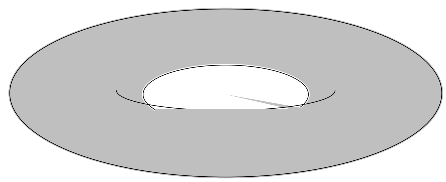


8. What if you can only make a certain number of cuts? Try cutting the circular cake in Problem 6 into eight same-size pieces with only three cuts. How many different ways can you find?

9. Can you cut the rectangular cake in Problem 5 into exactly twelve same-size pieces with only four cuts? Is there more than one way to do it?

How do the cuts in Problems 8–10 differ from the cuts in Problems 5–7?

10. What is the maximum number of pieces (they don't have to be same-sized) into which you can cut a bagel with three straight slices? (Ignore the crumbs!)



11. When a cake is frosted, the cutting task becomes harder. You want to make sure everyone gets the same amount of cake *and* frosting. What if the cakes from the earlier problems have frosting? How will you cut the cakes to give equal amounts of cake and frosting to each person?

Go back and try each of Problems 5–7 again, this time making sure each piece of cake has the same amount of frosting as well as cake. (Assume that the cakes are frosted on the top and all sides, but not on the bottom, and that the frosting is of the same thickness all over.)

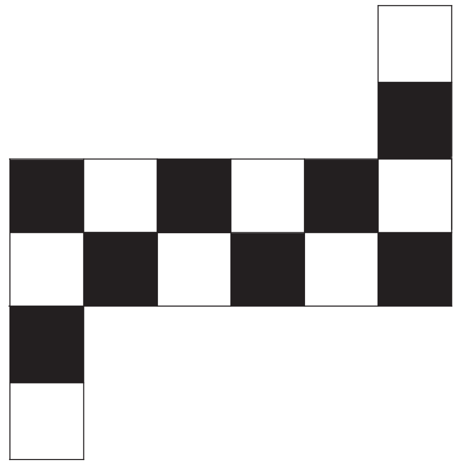
CHECKERBOARD PROBLEMS

Are checkerboards usually rectangular or square? What's the difference?

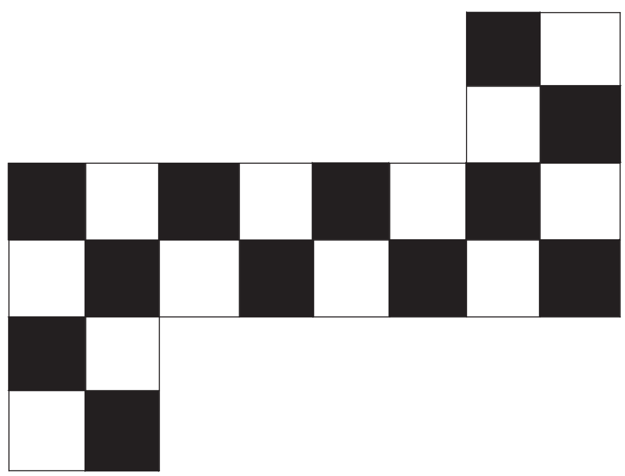
Checkerboards are . . . well, checkered; they have an alternating pattern of black and white (or black and red) squares. In these problems, someone has rearranged pieces of checkerboards into weird shapes. Figure out how to dissect the pieces into rectangular checkerboards.

"Same size and shape" sounds familiar . . . isn't that "congruent"? How can you be sure that the two pieces are "congruent"?

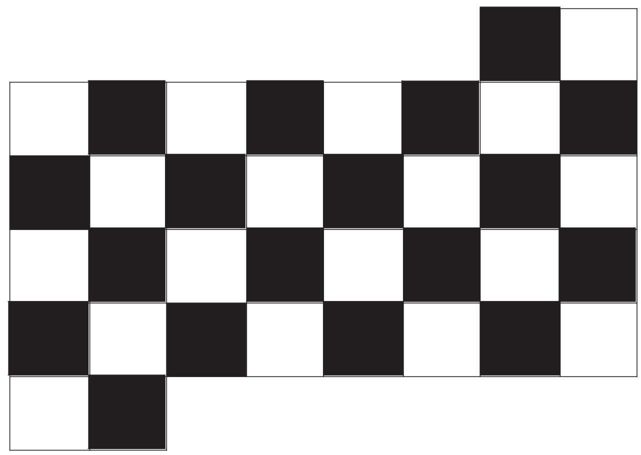
12. Copy this piece, and cut it into two pieces that can be put together to form a 4×4 checkerboard. The two pieces must be the same size and shape.



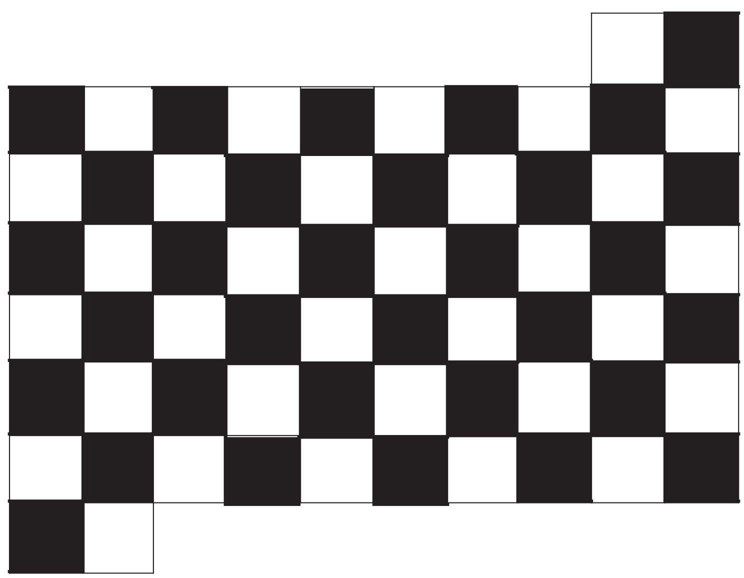
13. Copy this piece, and cut it into two pieces that can be put together to form a 4×6 checkerboard. The two pieces must be the same size and shape.



14. Copy this piece, and cut it into two pieces that can be put together to form a 6 × 6 checkerboard. The two pieces must be the same size and shape.



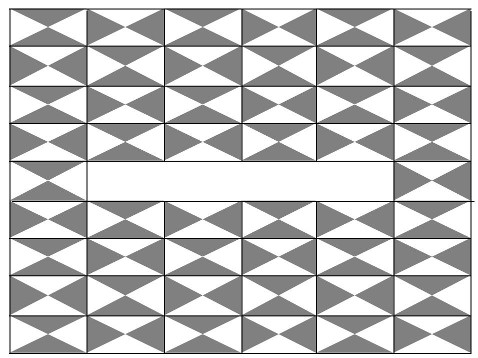
15. Copy this piece, and cut it into two pieces that can be put together to form a rectangular checkerboard. The two pieces must be the same size and shape.



RESTORE THE RUG

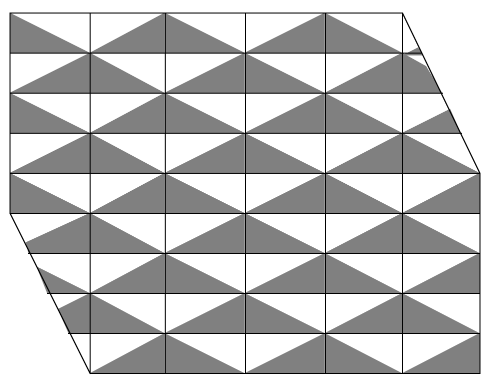
$9' \times 12'$ means "nine feet by twelve feet."

16. This $9' \times 12'$ rug has had a $1' \times 8'$ hole cut out of it where someone spilled cranberry juice. Instead of throwing away the rug, figure out how to cut it into two pieces that are congruent and that can be put together to form a rectangular rug with the original pattern but with no holes. With your own copy, make the cuts and put the rug back together. Remember: the two pieces must be the same size and shape, and the pattern must be the same.



You may want to make a few copies of the picture and test some ideas. This is not as easy as the earlier problems.

17. This $9' \times 12'$ rug has had two $2' \times 4'$ triangles cut from the corners where it has become worn. With your own copy, cut this rug into two congruent pieces that can be sewn together to make a square rug. Remember to repair and preserve the pattern of the black and white triangles.



18. Is it possible to cut a 9×16 rectangle into two congruent pieces, rearrange the two parts, and get a 12×12 rectangle? Explain.

Module Overview and Planning Guides

About the Module

This module focuses on the mathematical topics of area and perimeter, dissections, and Pythagorean Theorem. The module is problem-based. You will notice that the notes in this guide often refer to specific problems and the outcomes from students' investigations of them. The habits of mind focused on in this module include algorithmic thinking, explaining and proving, and using formal and informal mathematical language to describe things.

Dissections and writing dissection algorithms provide a context for students to learn about area and perimeter and to get some more experience with proof. Students who complete this module will gain extensive experience in using the Pythagorean Theorem and will develop both an algebraic and a geometric understanding (based on area) of the theorem. Several dissection/area proofs of the theorem are provided.

This module is ideal for:

- high-school geometry
- middle-school geometry (Use the "Hands-On Investigation" alternate timeline.)
- a course for preservice teachers

Main Timeline Planning Chart

Using all the investigations in this module will take an average class 6 to 9 weeks. Please note that Investigations 3.12–3.15 are meant as flexible resources. Plan to pull problems and activities from them regularly.

Investigation	Description	Key Content	Pacing
3.1 Tangram Activities	This investigation involves solving tangram puzzles. Students work in pairs for an informal, fun introduction to area concepts. The investigation bridges ideas about congruence developed in the module *A Perfect Match* and ideas about dissection and area, which are the focus of this module.	• dissection • area as invariant	2–3 days

Investigation	Description	Key Content	Pacing
3.2 Cut and Rearrange	This is a small-group activity. Students dissect a shape and rearrange the pieces to make another shape. Novel problems are presented that seem easy but engage students in discussion of area, shape, dissection, informal proof, and properties of the shapes.	• area • transformations • dissection	5–7 days
3.3 Cutting Algorithms	This is a partner activity. All students write algorithms for their cutting solutions. Partners debate why the shape they created not only looks like a rectangle, but *must be* a rectangle, given the cuts and transformations that were used in creating it.	• writing algorithms • debugging algorithms • quadrilaterals	2–3 days
3.4 Area Formulas	Students discover the connection between their cutting algorithms and area formulas. This investigation involves full-class discussion and problems. Students work on creating new formulas and making sense out of familiar ones.	• area of polygons • area of circle • the extreme case	4–6 days
3.5 More on the Midline Theorem	Many of the dissection solutions involve cutting along midlines. Here the properties of midlines, which have been informally applied earlier, are explored fully and proved.	• midlines	1 day
3.6 The Pythagorean Theorem	Students study one dissection proof of the theorem, read about the Pythagoreans, and solve a variety of problems applying the theorem. This investigation is suitable for any type of class organization.	• Pythagorean Theorem • visual proof • distance formula • Pythagorean triples	2–4 days
3.7 Changing Shape	Groups work on more advanced cutting problems. This investigation provides follow-up of Investigations 3.2 and 3.3.	• dissection • area	3–4 days

Investigation	Description	Key Content	Pacing
3.8 Equidecomposable Figures	This involves reading and explanation of a proof of the theorem: "If two rectilinear figures have the same area, then they are equidecomposable." It is challenging and thorough, with explanation of lemmas necessary for proof.	• proof, lemma • equidecomposability	1 day
3.9 Area and Perimeter	A square is dissected following a given algorithm, and then the dissection is repeated on the new figure. After each repetition, students look for a pattern in the changing perimeter.	• area and perimeter • recursion • isosceles right triangles	1–2 days
3.10 Making the Most of Perimeter	This is a study of a dissection proof of the following theorem: "Of all the rectangles with a given perimeter, the square has the most area."	• Isoperimetric Theorem • optimization	1–2 days
3.11 Analyzing Dissections	These dissection problems are suitable for individual or group assessment or projects.	• assessment and home-work problems	resource
3.12 Pythagorean Cutting Proofs	More visual proofs for students to explain are presented in this challenging investigation. Use them for projects, assessment, or extension.	• picture proofs • Pythagorean Theorem	optional 1–3 days
3.13 A Cutting Paradox	Students try to explain why a simple dissection seems to work but doesn't. The explanation involves a side trip into the world of Fibonacci numbers. This serves as a warning to reason carefully about dissection results. This material is suitable for projects, assessment, or extension.	• Fibonacci numbers • dissection that doesn't work	optional, 1–2 days
3.14 Cutting Up Solids	Students create cross sections for cubes, spheres, and cones and discuss which solids can be dissected and reformed into other solids. A short historical essay about Hilbert problems is included. Use for projects, assessment, or extension.	• cross sections • volume, surface area • Hilbert problems	optional 2–3 days
3.15 Cut a Rug and Other Dissections	These are fun diversions. Students make a witch's hat, cut up a checkerboard, or dissect a birthday cake.	• fair division • development of cone	optional 1–5 days

ALTERNATE TIMELINES

We offer two alternative paths through the module, each with a specific emphasis and all considerably shorter than the main timeline plan. If the full 6–9 weeks is too long for your class, try one of these. These alternatives were chosen to meet the most frequent requests of field-test teachers.

Hands-On

- **3.1 (3 days)**
- **3.2 (7 days)**
- **3.3 (3 days)**
- **3.4 (4 days)**
- **3.6 (3 days)**
- **Optional selections from 3.7, 3.9, and 3.12–3.15**

Hands-On Investigation (4–5 weeks)

Students study dissection, area, and the Pythagorean Theorem in this highly exploratory, hands-on unit. Proof is kept at an informal level. This timeline is a good choice if you are looking for an *unusual* supplement to a traditional text, if yours is a middle-school class, or if your curriculum doesn't emphasize formal proof.

Proof-Focused

- **3.2 (5 days)**
- **3.3 (2 days)**
- **3.4 (4 days)**
- **3.5 (1 day)**
- **3.6 (3 days)**
- **3.7 (3 days)**
- **3.8 (2 days)**
- **Optional selections from 3.9, 3.10, 3.13, and 3.14**

Focus on Proof (4–5 weeks)

This timeline is for those who wish to take the fullest possible advantage of the module's deep and wide offerings on proof while de-emphasizing the hands-on activities. Note that your class should be familiar with the notion of congruence in order to complete many of the proofs in this module.

You will move quickly through the cutting problems and creating algorithms, focusing later on proofs of the algorithms, the Midline Theorem, and the Pythagorean Theorem. Investigation 3.8, on equidecomposable figures, leads students through an interesting and difficult proof. It is an ideal capstone to this proof-focused timeline.

ASSESSMENT PLANNING

Expect students to make about three group presentations, to take several individual written quizzes, and to submit a portfolio at the end of the module. The presentations will be part of class routine in this module, but plan to grade them more formally. Oral presentations should be accompanied by written solutions for easier assessment. The quizzes will be based primarily on "Checkpoint" problems and may be in-class or at-home quizzes.

What to Assess

- The student can read, write, and "debug" algorithms.
- The student has developed problem solving skills (especially mixing deduction with experimentation).
- The student understands the proof and application of the Pythagorean Theorem.
- The student has improved visualization skills (in both 2 and 3 dimensions).
- The student understands how area formulas are related to properties of the figures and can develop a method to find the area of a polygon by using dissection.
- The student uses geometric vocabulary correctly in proofs and explanations, including: *bisect*, *perpendicular*, *altitude*, *median*, *midline*, *parallel*, *parallelogram*, *rectangle*, *square*, *rhombus*, *isosceles*, *equilateral*, *scalene*, *perpendicular bisector*, *tetrahedron*, *cube*, and *equidecomposable*.
- The student can demonstrate knowledge of dissection, area, and polygonal figures.

Notebooks

Throughout the entire module, we recommend that students keep a notebook containing:

- daily homework and other written assignments
- a list of vocabulary, definitions and theorems that emerge during classwork and homework

Portfolios

This collection of work should be submitted at the end of the module. Announce the requirements for the portfolio contents when you begin the module. We suggest that portfolios include most of the following:

- the list of vocabulary, conjectures, and theorems that the student has kept (You may want students to organize and refine the list before putting it in the portfolio so that it becomes a summary of the student's learning.)
- samples of the student's best or favorite work
- a reflective essay about the student's learning in the module
- the solution to a new dissection problem (chosen from the problems in Investigation 3.11)

If the class has spent a great deal of time in the module, or if you want students to take on a more extensive piece of individual work, the portfolio could include a final project, such as:

- one of the Investigations 3.12–3.15
- Problems 18, 20, and 21 from Investigation 3.7
- the square dissection problem in Investigation 3.9
- *Take It Further* Problems 5 and 7 from Investigation 3.10

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
3.2	Give group presentation of solution to dissection problem.	*Checkpoint* Problem 16: Explain a given trapezoid dissection.
3.3		*Checkpoint* Problems 13 and 14: Supplement with a dialog assessment as in Problems 1–3 or Problem 8 from Investigation 3.11.
3.4	Explain how a formula can be derived from dissection.	Use the optional challenge quiz suggested in the Teaching Notes.
3.6		Give a teacher-generated content quiz based on applications of the Pythagorean Theorem.
3.7	Give presentation of algorithms for dissection.	
3.11	Gives presentation of assessment problems from Investigation 3.11.	

Teaching Notes

Investigation **3.1**

Student Pages 1–9

TANGRAM ACTIVITIES

Materials:
- **Tangrams (one set per student)**
- **Puzzle problems**

See "Additional Resources" for information about tangram sets.

Technology: Geometry software is optional. See "Using Technology."

OVERVIEW

The first section of the module focuses on the technique of dissection—cutting up a figure into pieces and then rearranging the pieces to form a new figure—as a forum in which to explore mathematical topics including

- creating algorithms (to dissect any triangle into a rectangle, for example);
- area formulas (rectangles, triangles, parallelograms, trapezoids, other regular polygons, and circles);
- properties of quadrilaterals (how do you know you have a parallelogram?);
- area and perimeter;
- the Pythagorean Theorem.

For Discussion *(Student page 1)* In everyday language, the word *area* has several meanings. When you say, "the area of the room is quite big," you are referring to something very close to the mathematical meaning of the word, but if you say, "I'll meet you in that area by the lake," you mean something quite different.

At this point, a good definition of *area* in mathematical terms might be the following:

> The area of a two-dimensional figure is a measure of how much space it takes up.

A more precise definition might be:

> The area of a figure is the number of one-unit squares which fit inside it, labeled in square units.

A major theme in this investigation is that two figures have the same area if they can be dissected into the same smaller pieces. In other words, if two different-looking shapes are created from the same two pieces, these two figures will have the same area, even though they may not be congruent.

Tangram problems provide a nice transition from the ideas of congruence to the ideas of area, shape, dissection, and transformation. The primary goals of the investigation are to begin talking about these ideas and to enjoy the challenge of the puzzles. While students discuss and present their solutions there will be opportunities to refine and review definitions and to practice clear communication. Encourage discussion of ways to tell if two shapes have the same area.

Students work in pairs or small groups for one class. Each student will need a set of tangrams and a set of the puzzles (use blackline masters to make these). Keep one set of tangrams for use with the overhead projector.

TEACHING THE INVESTIGATION . . . .

The introductory discussion is meant to serve as a pre-assessment of students' existing ideas about area. For preview homework, ask students to write answers to the discussion questions in their journals. Then, when you begin class, use their written ideas to fuel the discussion.

Have a student read the "Perspective on the History of the Tangram" for the class as a way to introduce the tangrams.

Introduce the tangrams after going over the homework definitions of area. Most of the class will be spent working on the puzzles and problems. At one or more points during the class, students should present their solutions on the overhead projector. These presentations offer a good opportunity to model and reinforce correct use of the geometric vocabulary in the investigation, but don't overload the activity with content. It is meant to be an opening of the conversation about area, not a full theoretical treatment of all the mathematics that will come up. Reasonable goals include:

- reinforcing the idea that cutting and rearranging shapes does not change area;
- introduction of vocabulary: rotation, reflection, translation;
- definition of area.

It is a good idea to warn the students about which problems they will be presenting before they start working on the puzzles. They might need to write down their solutions in order to remember them.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

- Use the "Checkpoint" or "Write and Reflect" problems as homework.
- Assign reading of "Perspective on the History of the Tangram" as homework.
- The "Take It Further" problem is challenging and would make an interesting homework assignment or project.

USING TECHNOLOGY ....

This investigation was not designed with the computer in mind, but a class with a particular emphasis on computer use could try the following problem using geometry software:

Look at the section "Moving the Pieces" in the Student Module. Using a computer geometry drawing tool, construct all three of the transformations that are pictured. (For example, start with a parallelogram, divide it into two triangles, and create a square by using the computer's translation tool.)

After you have a good idea how the shapes in the text were created, answer this question: How many other shapes could be created in this same way? To be more specific:

Decide as a class what will count as a "shape." Does it have to be a polygon? Can it have more than four sides?

- First, start with a parallelogram; divide it into two triangles by drawing a diagonal.
- Next, try to create a new shape by rotating, reflecting, or translating one of the triangles.
- How many *different* shapes can be made from the parallelogram in this way?

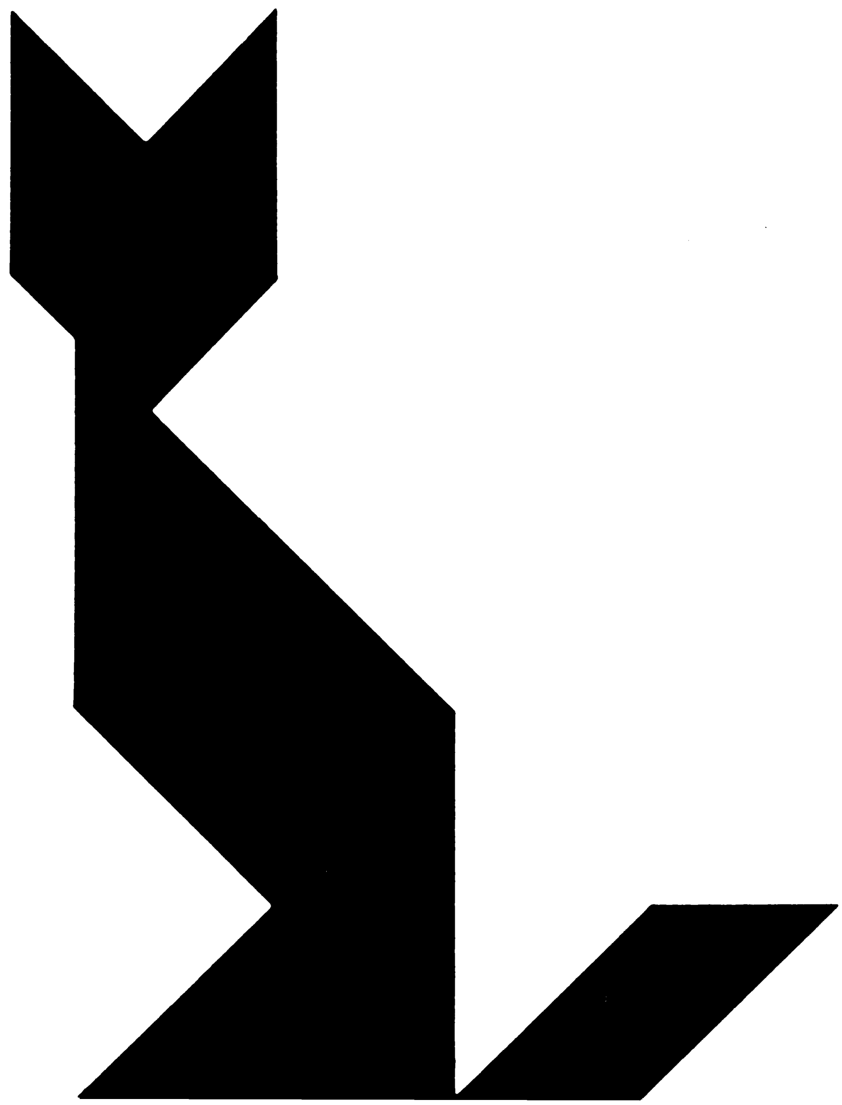
ADDITIONAL RESOURCES ....

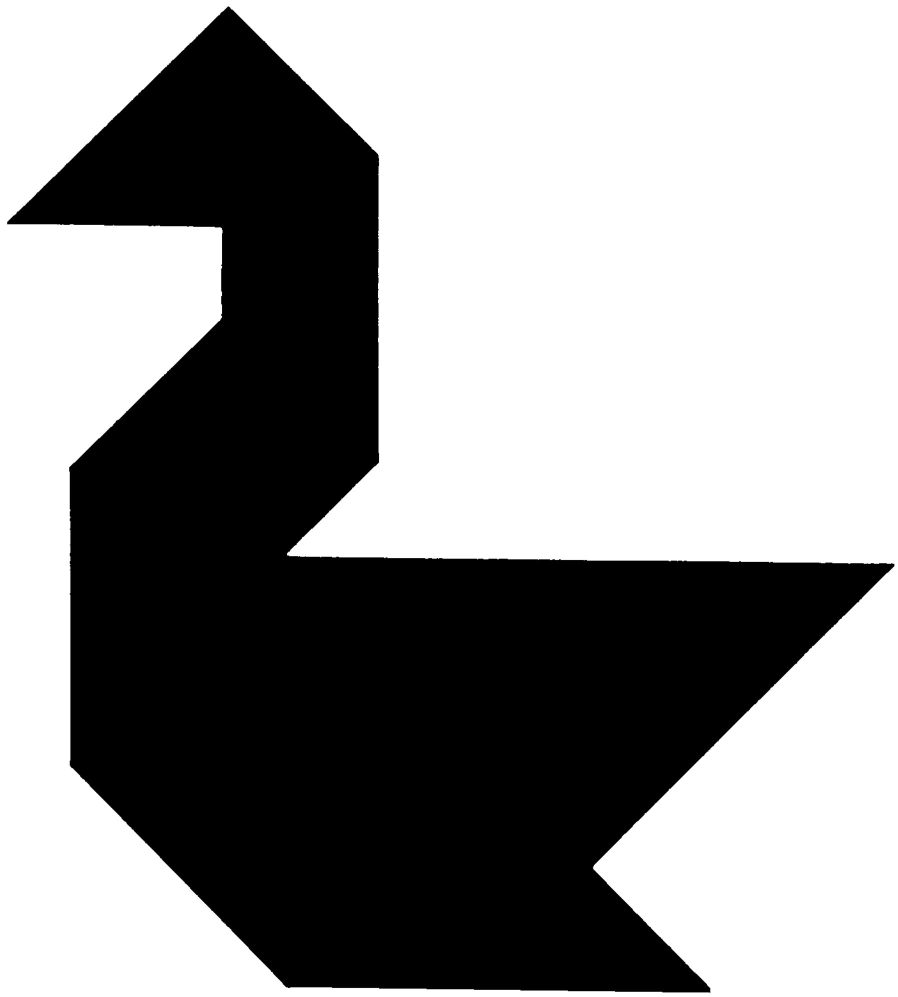
A benefit of having students create their own sets is that they then have personal tangram sets to keep and to take home. A downside is that Problem 3 is no longer a challenging problem for any student who remembers how they made the set.

Tangram sets, including sets that are transparent for use on an overhead projector, are available from most publishers of educational manipulatives. Students can also make their own sets out of cardboard or stiff paper. A blackline master of the dissection of the square into tangram pieces is included after the tangram puzzles. This can be given to students to cut out.

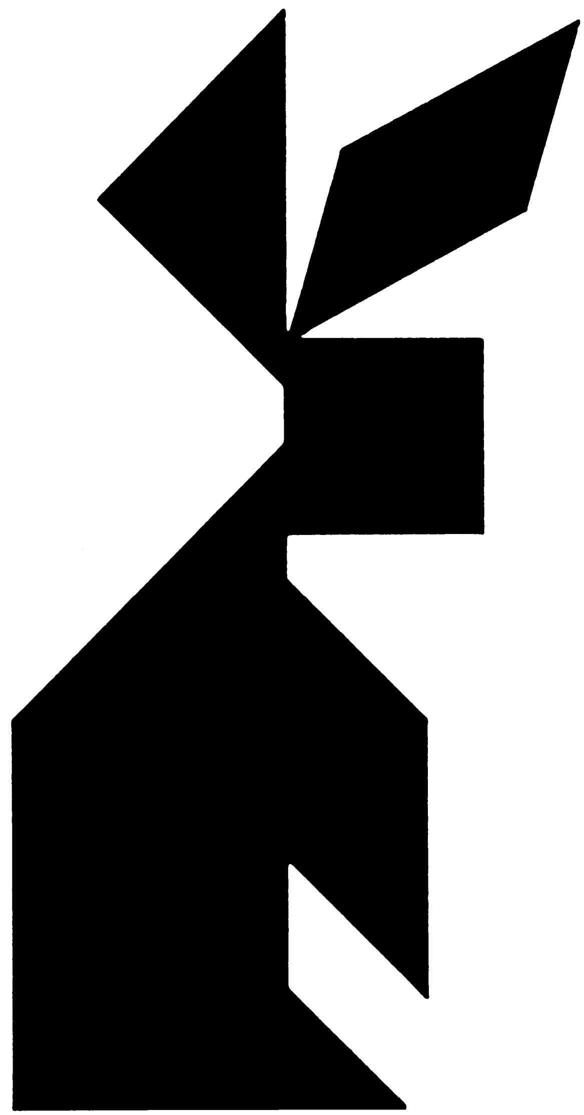
One teacher decided to make a problem out of it: She gave the dissection to one student and asked her to write directions for the other students to follow in order to complete the dissection. For example, she said, "Cut along the diagonal of your square. Now take one of the triangles you created, and connect the midpoint of the longest side to the other vertex. Cut along that line. Keep those two triangles." The students had to listen carefully to the directions and follow them as they were read in order to construct their own tangram sets.

TANGRAM ACTIVITIES *(continued)*

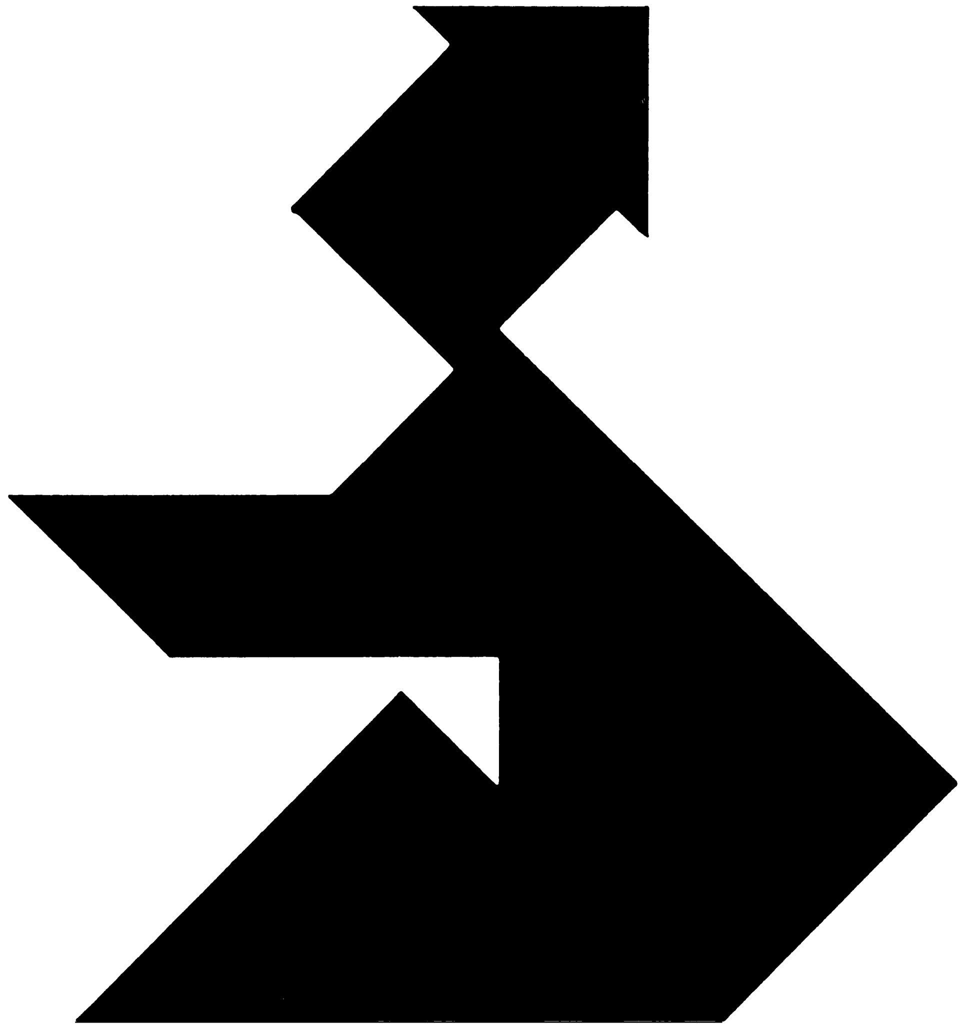




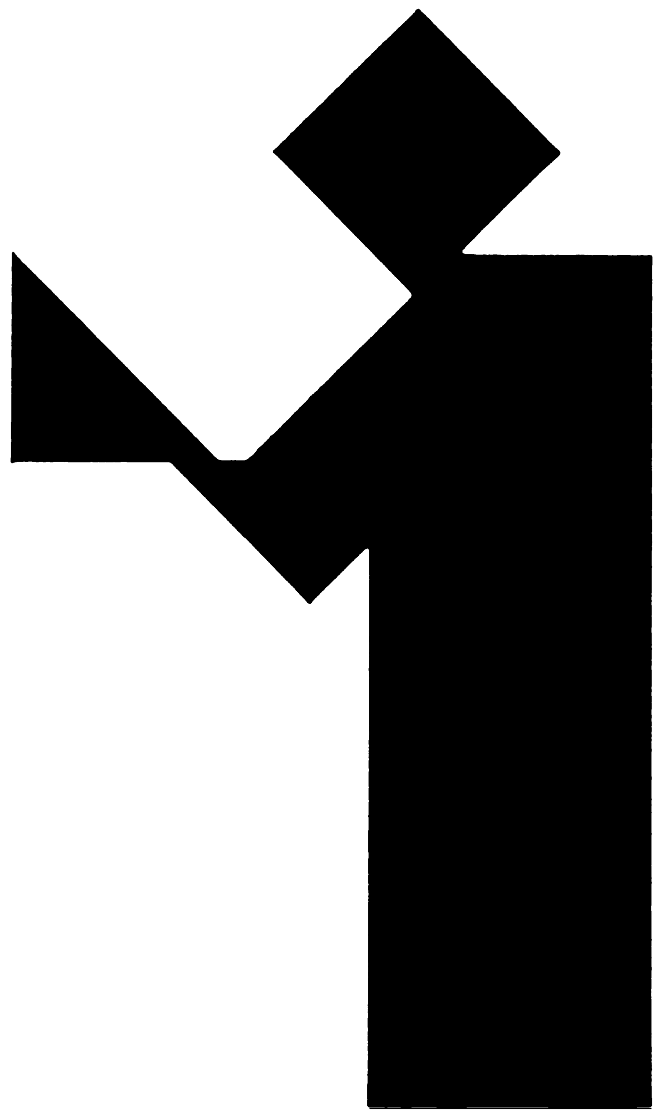
Tangram Activities *(continued)*



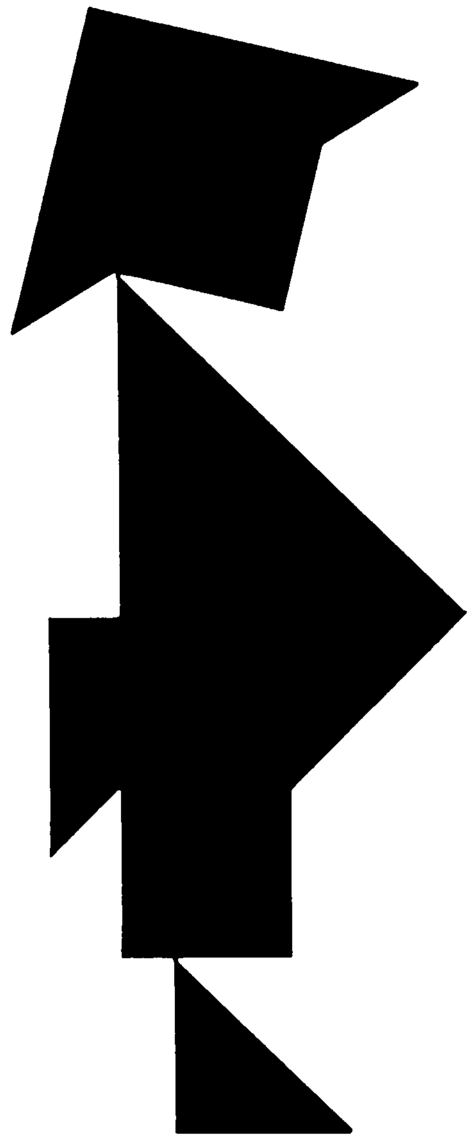
TANGRAM ACTIVITIES *(continued)*



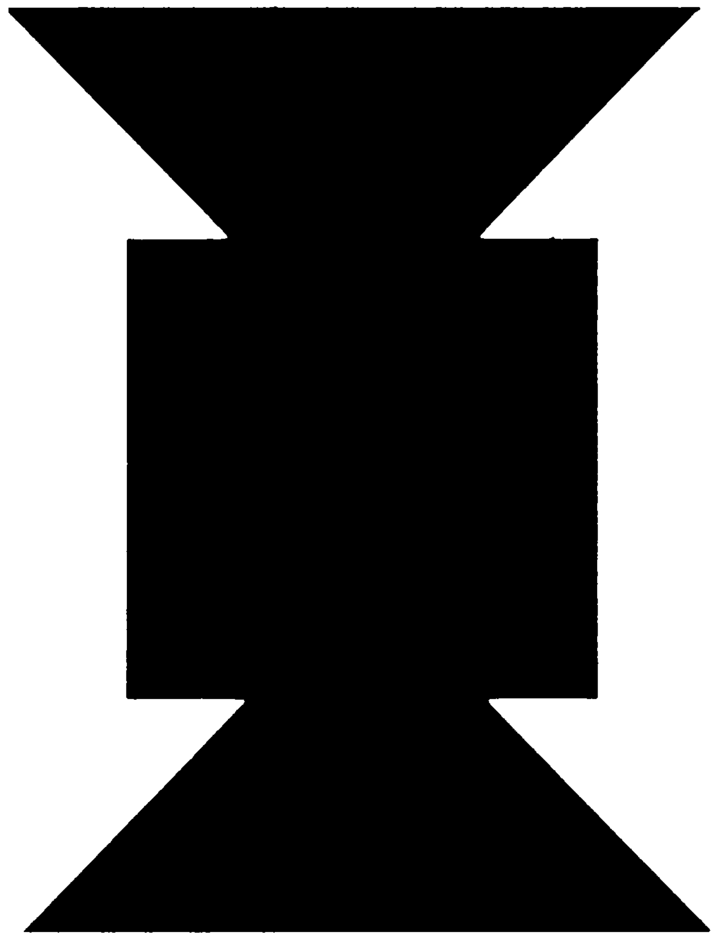
TANGRAM ACTIVITIES *(continued)*



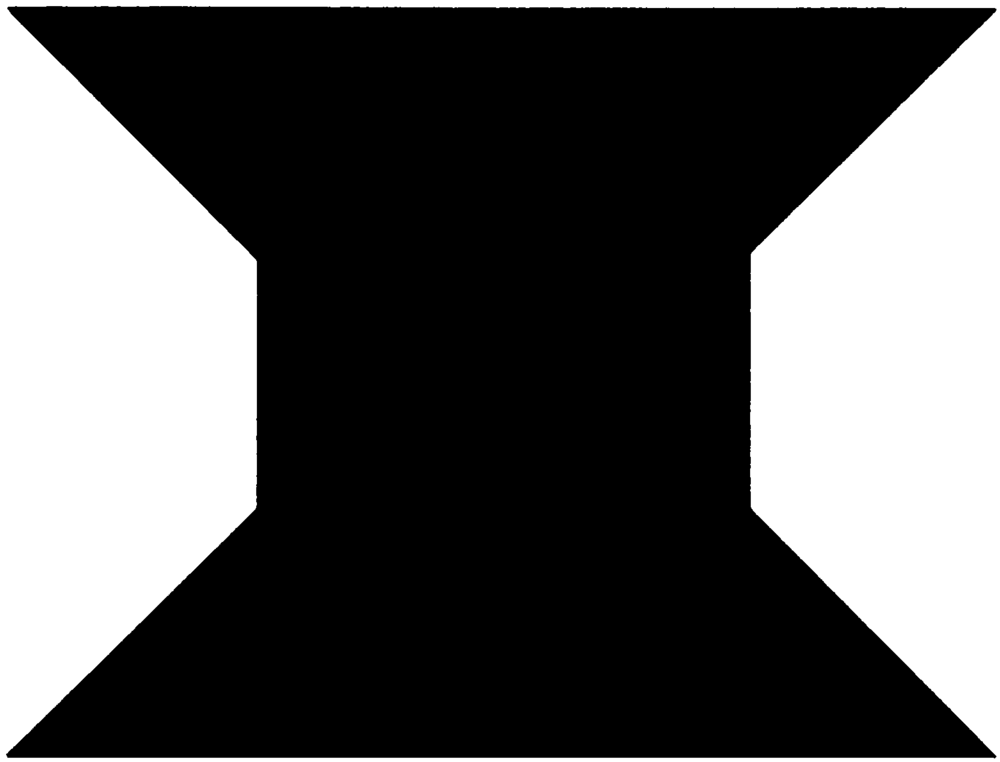
TANGRAM ACTIVITIES *(continued)*



TANGRAM ACTIVITIES (continued)

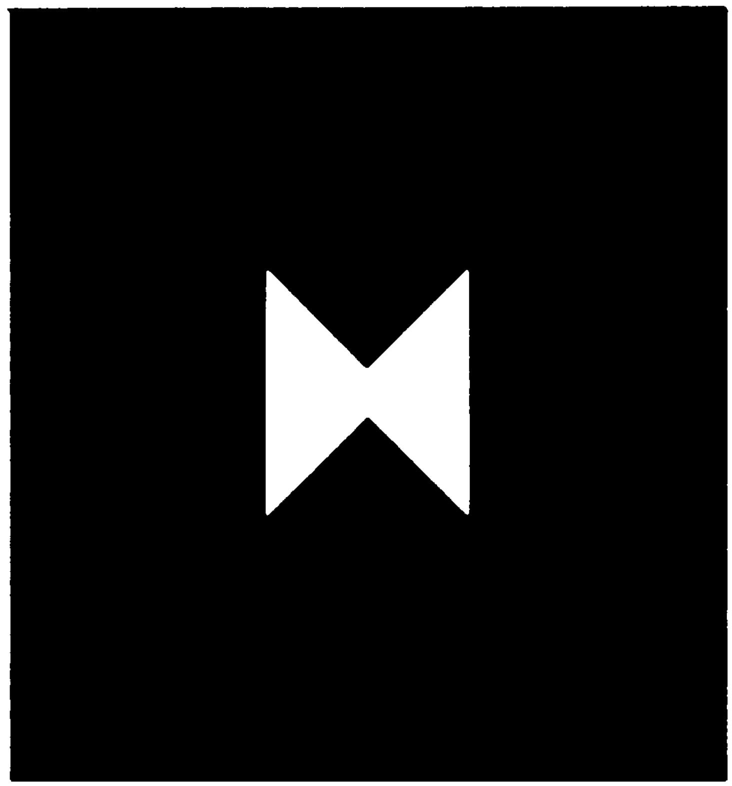


TANGRAM ACTIVITIES *(continued)*

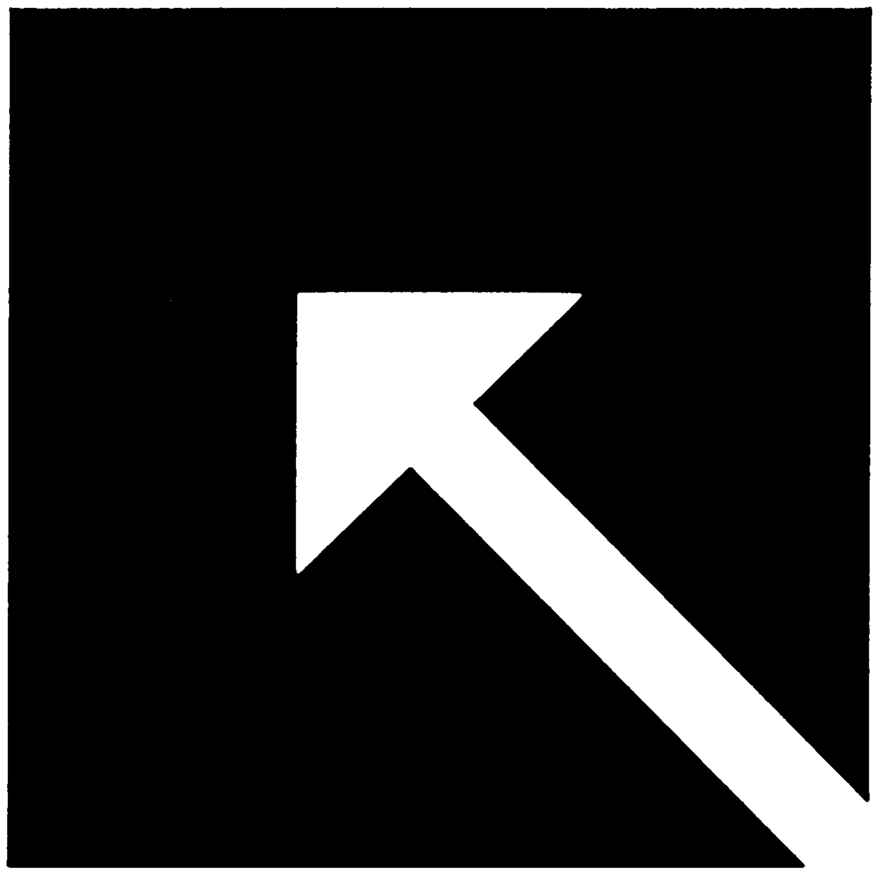


TANGRAM ACTIVITIES *(continued)*

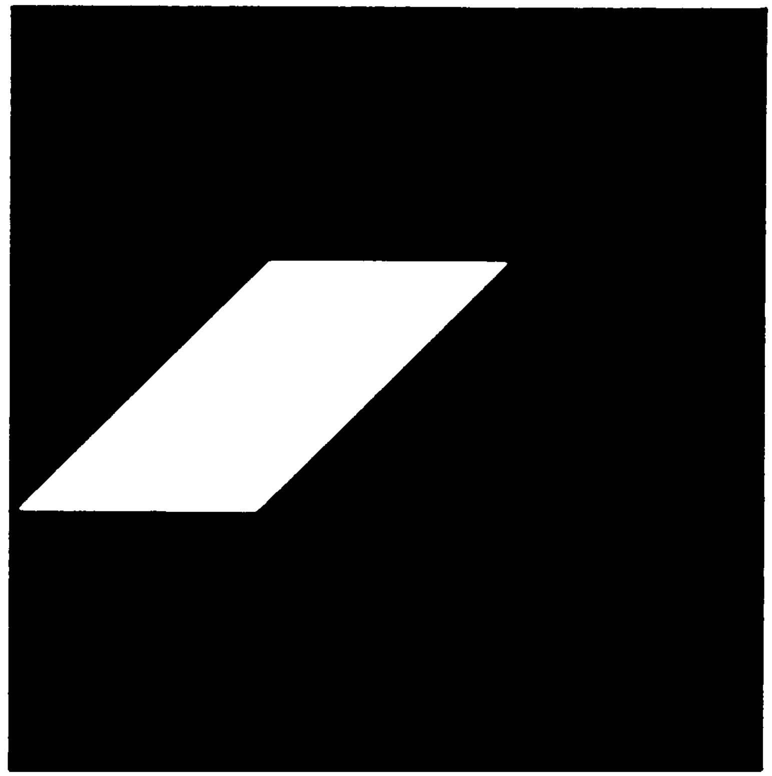
TANGRAM ACTIVITIES (continued)



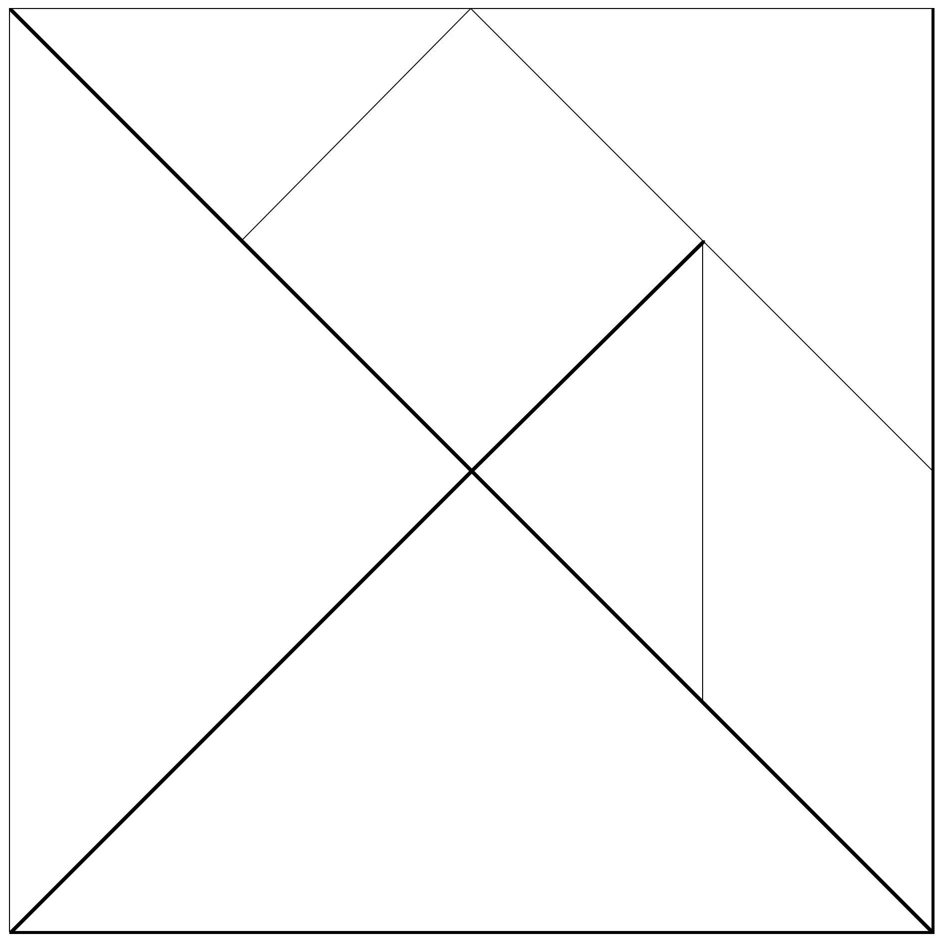
TANGRAM ACTIVITIES *(continued)*



TANGRAM ACTIVITIES *(continued)*



Dissection of a Square into Tangram Pieces



Teaching Notes

Investigation **3.2**

Student Pages 10–22

Cut and Rearrange

Overview

Materials: Provide each group with a Cutting Kit containing scissors, rulers, and lots of paper. This kit will be needed almost daily for some time.

In this investigation, students will be solving dissection problems like: "Cut up a triangle and rearrange the pieces to form a rectangle." To a casual visitor, the classroom might seem to be full of art students making collages. But, as the problems are solved and explained, the mathematics involved becomes clear and is extensive. To defend their solutions, students will need to understand properties of parallelograms, kites, trapezoids, rectangles, rhombi, and squares, and to develop general classifications for triangles and quadrilaterals. Emerging ideas will include the concept of a relation, measure as a geometric invariant, dissection, and the introduction of a relation called "scissors-congruent" that is equivalent to "has the same area as."

The novelty of the problems encourages students to develop new problem-solving strategies and habits of mind. These include using proof as a research technique and a method of persuasion, using algorithms as arguments, and using argument as a method for refining ideas.

Each student or group must have scissors, straightedges, and a supply of paper for cutting. Copies of the problem figures can be made ahead by the teacher or traced by the students, but must be provided in some form that actually allows cutting. Making an overhead transparency of each figure will facilitate student explanations.

Hint: If the original copy of each problem figure is on colored paper, and all of the cutting is done with white paper, then the students will never lose the original or accidentally cut it up.

One teacher reports: "Students found the problems engaging. Finding solutions allowed creativity and experimentation, but defending the solutions demanded some fairly rigorous thinking. There was plenty of traditional 'content.' The mixture of concrete and abstract was very stimulating for class discussion. Students have a whole new concept about shapes now, and keep bringing it up now that the unit is over."

Students are likely to come up with a variety of different and creative solutions for each of these problems. Be sure to ask, "Did anyone do it a different way?"

Teaching the Investigation ....

Day	Discussion	Homework
Day 1	Work on Problems 1 and 2; present some solutions in class.	Problems 3 and 4. Students should be ready to demonstrate their solutions.
Day 2	Students present homework solutions and discuss what makes the solutions really "work." Have groups do Problems 5 and 6; present solutions if there is time.	Problem 7
Day 3	Discuss Problems 5–7. Class discussion about Problem 8. Work in groups on Problems 9–12.	Problems 9–12. Students be ready to present solutions.
Day 4	Discuss solutions to homework and Problem 13. Groups work on Problems 14–15.	Redo solutions to Problems 1–6, refining the cutting processes and including pictures and explanations. This is a final assessment and should be presented as such.
Day 5	Collect homework. Students work on "Checkpoint" and "Take It Further" problems.	

All of the problems in this investigation involve cutting and rearranging geometric shapes to produce other shapes that are equal in area. The relation "equal in area" is a theme that dominates the remainder of the module, so it is important to have an initial discussion of what it means and some agreement by the class on what term they will use to describe it.

CUT AND REARRANGE *(continued)*

After this discussion, break the class into groups to work on the first problem. (Cut a parallelogram into a scissors-congruent rectangle.) The groups are likely to produce several good solutions to this problem fairly quickly. Have a few students share their solutions with the whole class. (Cutouts can be placed directly on the overhead projector.) Once they are reassured that these *weird* problems really do have solutions, and that there can be more than one right answer, students should return to group work and try more problems.

The amount of time it takes to do the rest of the problems can vary greatly from class to class. It might take some classes a whole period to be successful with the first problem. One teacher spent a full day on each cutting problem, taking nearly two weeks to complete this investigation. She reported it as a highly successful investigation, though, because the students felt a sense of accomplishment with their growing competence, and because they learned so much mathematics as they were working.

However you structure the time to be spent on this investigation, keep in mind that it is rich in both problem solving and mathematical content. Let it develop at a pace that makes sense for the needs of your class. Plan to spend a lot of class time, especially in the beginning, having students demonstrate and explain their solutions. Several students should present solutions to each of Problems 1–7, and the solutions should be discussed.

In the Student Module, we define *scissors-congruent*:

> Figure A is *scissors-congruent* to figure B if A can be cut up and the pieces rearranged to form figure B.

One class chose to use *covers* instead of *scissors-congruent* because they felt that it described the test they were performing. We chose not to use “covers” in the Student Module because it has other mathematical meanings, but would encourage teachers to discuss with students possible names for this relation, and to support their efforts at making definitions and inventing language to describe mathematical ideas.

CUT AND REARRANGE *(continued)*

Here are some things to remember as you are teaching the class:

The first day can be very frustrating to many students who don't know *how to do* these problems. But by day 2 or 3, there is usually plenty of success.

- Many students will have no idea how to do the problems, and they may begin by cutting almost randomly or begin with very awkward strategies. Don't give too many hints. The struggle pays off.
- Remind students that they must use *all* of the cut up pieces in making the new shape so that the two figures are "scissors-congruent."
- While students are working in groups, ask a lot of questions like: "What do you think might work?" "How do you know that idea won't work?" "How do you know the shape you made is really a rectangle?" "Why did you make the cut right there?"

A nice activity that reinforces the need for care and reasoning in cutting is *A Cutting Paradox* (Investigation 3.13).

What's Coming Up? In the next investigation, students will be writing algorithms for the solutions to the problems in this investigation, so be sure they save their notes!

- When students are presenting their solutions to the whole class, have them describe their method of solving the problem and explain *why* it works. Steer them away from vague "it looked right" statements.
- At this point, if the solutions work for the particular figure in the problem, they should be considered correct. In the next investigation, students will be formalizing their solutions from this investigation into general methods that would work in all cases.
- Students may say, "I cut in the middle . . . " when explaining their solutions. You can ask clarifying questions like, "What do you mean by 'middle'? Do you mean at the midpoint?" Soon, the correct vocabulary will become a part of the classroom culture for explaining solutions. Students will communicate more clearly in order for you to understand them and for them to understand each other.

ASSESSMENT AND HOMEWORK IDEAS.......

When students arrive at class, they should be able to show you homework. You won't necessarily collect it each day, but you will look at it, and they are expected to share solutions when asked.

Homework ideas are suggested above in "Teaching the Investigation." Almost everything the students do in this investigation could be used for assessment. For example:

- Near the end of the investigation, collect and grade written solutions to Problems 1–6. Combining drawings with explanatory text is an excellent exercise in communicating about mathematics.
- Sit in the back of the room while the students present their solutions and assign points for items on an assessment checklist that you have prepared in advance and shared with your students. You could include correctness of solution, clarity of explanation, understanding of area concepts, correct use of certain vocabulary words, effectiveness of problem solving strategy, and communication skills.
- Assign the "Take It Further" or "Checkpoint" problems as homework or assessment problems.
- Have the class select the most elegant or the most efficient solution to a problem that everyone worked on for homework. Each student could write about why they think a particular solution has merit.
- There will be a great deal of vocabulary flying about during these lessons. Quizzing on some of these terms would be appropriate.

For this investigation and the next, choose assessments so that some are written and some oral, some are individual and some group.

The original article was Margaret Biggerstaff, Barb Halloran, and Carolyn Serrano, "Teacher to Teacher: Use Color to Assess Mathematics Problem Solving." *Arithmetic Teacher* 41 (February 1994), pp. 307–8.

Carol Martignette-Boswell, a teacher from Arlington, MA, decided to adapt an assessment technique from NCTM's journal *Arithmetic Teacher* and incorporate peer assessment while working through this book. After a group of students presented a solution to a cutting problem, the other groups would rate the presentation using colored index cards, each of which had criteria written on the back:

White card: 0 points. Incorrect solutions with no explanation.

- Unable to cut the pieces and complete a rearrangement.
- Unable to give dimensions of the new shape in terms of the original shape using mathematical language.
- Unable to describe the process used to make the rearrangement.
- Unable to present an argument.

Cut and Rearrange (continued)

Green card: 1 point. Started on a solution but used an unsuitable strategy.

- Able to cut the pieces and make a rearrangement.
- Unable to use mathematical language to give dimensions of the new shape in terms of the original shape.
- Unable to describe the process used to make the rearrangement.
- Unable to present an argument.

Yellow card: 2 points. Used a reasonable strategy but did not finish or reach a solution.

- Able to cut the pices and complete the rearrangement.
- Used some mathematical language to give dimensions of the new shape in terms of the original shape.
- Unable to describe the process used to make the rearrangement.
- Unable to present an argument.

Red card: 3 points. Used a reasonable strategy to reach the correct solution but gave an unclear explanation.

- Able to cut the pieces and complete the rearrangement.
- Able to use mathematical language to give dimensions of the new shape in terms of the original shape.
- Able to describe the process used to make the rearrangement.
- Unable to present an argument.

Blue card: 4 points. Used a reasonable strategy to reach a correct solution and give a clear explanation.

- Able to cut the pieces and complete the rearrangement.
- Able to use mathematical language to give dimensions of the new shape in terms of the original shape.
- Able to describe the process used to make the rearrangement.
- Able to present an argument.

Of course there were discrepencies in the scoring, but one group never scored a presentation “white” while another group scored it “blue”; the students found that they really *did* know what made a good explanation.

CUTTING ALGORITHMS

Materials:
- **Cutting Kits**
- **transparencies for demonstrations**
- **students' solution notes and property lists from Investigation 3.2**

Technology: Geometry software and Logo are both optional. See "Using Technology."

OVERVIEW

In the previous investigation, students were exploring, looking for solutions, and defending conjectures. The outcome was a reasonable explanation for a correct dissection of a particular figure. Here the emphasis shifts to algorithmic thinking, systematic testing, critical methods of analysis, and deductive reasoning. The outcome will be general algorithms which describe correct dissections for a whole class of problems. (Students will create, for example, an algorithm for cutting *any* triangle and rearranging the pieces into a rectangle.) Three main threads run through this investigation:

- understanding what algorithms are, and learning to write them;
- understanding what is meant by a *standard* case and an *extreme* case, and how these apply to the testing of algorithms;
- continued exploration of area and of the determining properties of polygons.

This investigation aims to help students become more sophisticated in their understanding of the rigor involved in classifying shapes and in distinguishing between the general and the specific case.

Students need to have completed at least Problems 1–6, 8, and 13 from Investigation 3.2. Be sure students bring their solution notes and the list of properties of the rectangle that they developed in Problem 13 to class.

TEACHING THE INVESTIGATION

This investigation works well by doing a partner activity between two full-class discussions:

Students might be more familiar with other words for this idea, like "recipe," "instructions," or "step-by-step plan."

First discussion: What is an algorithm? Students have had a lot of experience with algorithms and can probably offer many good examples once they understand the meaning of the word.

Activity: Students work with partners to write and test algorithms. The final product is refined algorithms for the three major cutting problems in the previous activity.

Second discussion: How do we check an algorithm and test it on nonstandard cases? Students study and critique the examples in the Student Module and refine their own algorithms.

Close the investigation with the writing assignment about what it takes to make a rectangle (Problems 9 and 10), or by having students make the final rewrite of their algorithms as suggested in the "Write and Reflect" Problem 11.

Students are asked to write three algorithms, test them, rewrite, discuss, and rewrite one last time. To save time, consider switching to groups after the partner activity, and assign each group *one* of the algorithms to process through the next stages of discussion and writing.

Looking for patterns or invariants: Are there certain cuts or "moves" that seem to occur in many of the dissection algorithms? Ask students to look out for these and to think of names for them (for example a "midline cut" or a "diagonal reflection"). Once named, these procedures could become part of the accepted classroom vocabulary for assignments and presentations.

Many students have had little experience with the sort of rigorous thinking required by this investigation. Such students may be content accepting a good solution for one case as a suitable solution for all cases that look vaguely similar in their eyes.

For Discussion *(Student page 28)* There are at least two natural approaches to use when trying to find a special case or counterexample. One is to study the given algorithm, and see if you can find places where it might break down. This works well for Problem 6; you can see in the given picture that the altitude must bisect the opposite side, and ask yourself if that is, in fact, true.

Another approach is to study some extreme figures, not the standard ones you normally imagine. For example, in Problem 7, you can begin by drawing many different types of parallelograms, being sure to include some oddly-shaped ones. It is these unusual figures which might cause an algorithm to fail. For this reason, it is important to keep these figures in mind when writing your own algorithms.

ASSESSMENT AND HOMEWORK IDEAS.......

- Use the "Checkpoint" problems as an in-class test or as a graded homework.
- The write-and-trade-algorithms partner activity is a powerful self-assessment for students. To increase its impact, ask students to write about how their algorithms changed or improved, or have them keep a complete set of rewrites for one algorithm and explain why each one was an improvement.
- Quiz material: properties of triangles and quadrilaterals, definition of an algorithm, successful reading of an algorithm.
- Try Problems 16, 18, or 19 in the "Take It Further" section as a challenging individual quiz or a good group quiz.

- For homework, have students write an algorithm for something from another context: for example, an algorithm for adding two digit numbers, for putting four names in alphabetical order, or for constructing an equilateral triangle.

USING TECHNOLOGY

A *script* or *macro* is a list of instructions saved on the computer. If you "play the script" or "run the macro" the computer will carry out the full list of instructions. Some interesting scripts are generally included with the software.

The main focus of this investigation is writing and checking algorithms, something computer programmers do all day long. To redirect this activity into the computer lab, use this as an opportunity to teach students how to use the *script* feature of the geometry software being used in the class. Then have them read and write dissection algorithms in the form of scripts. For example:

- Write and test a script that will dissect a given parallelogram and rearrange the parts into a rectangle.
- Study a script that is already written; then imagine and draw the construction you think it will make. Compare your drawing to what the script actually does on the computer.

Writing algorithms (programs) in Logo also makes sense. For example:

- Write a Logo program to construct a trapezoid (or parallelogram). Explain why it produces a figure with all of the correct properties.
- Write a Logo program that will draw a parallelogram that is marked with the cutting lines needed to dissect it into a scissors-congruent rectangle. Explain why the Logo commands produce the correct dissection marks.

Teaching Notes

Investigation **3.4**

Student Pages 32–49

AREA FORMULAS

Materials:
- **Cutting Kits**
- **cutting algorithms from Investigation 3.3**

Technology: Geometry software is optional. See "Using Technology."

OVERVIEW

In this investigation, students develop area formulas for the parallelogram, triangle, trapezoid, and circle. They have already found ways to take a triangle and make it into a rectangle; they have formalized the cuts and rearrangement into algorithms and have explained why the algorithms work. Now, they will compare the dimensions of the triangle with its scissors-congruent rectangle, expressing the dimensions of the rectangle in terms of the base and height of the triangle. The area formula for the triangle emerges as students express the area of the scissors-congruent rectangle. (If the triangle's base is b and height h, then the rectangle's base is b and its height is $\frac{1}{2}h$, giving both figures the area $\frac{1}{2}bh$.)

What we are striving for in this investigation is for students to see that they can develop their own formulas for area from their own cutting algorithms. If their past experience with formulas has been memorization and application, this activity could be a real "Aha!" experience. One student exclaimed, "So *that's* where the formula came from!"

Completion of Investigation 3.3 is necessary for understanding this investigation because students will use the algorithms they have developed to create their own area formulas.

The area formulas developed here all depend on students' knowledge that the area of a rectangle is base $\times$ height (or length $\times$ width). If you're not sure that your students remember this, you may want to begin the first day with a short review about the area of rectangles.

TEACHING THE INVESTIGATION

In this investigation, we are asking students to assign variables to represent the base(s) and height of an uncut figure (a triangle, parallelogram, or trapezoid), to cut and rearrange the figure into a rectangle, and then to express the base and height of the rectangle in terms of the dimensions of the original uncut figure. This requires a degree of abstraction that may be confusing and difficult for some students. Plan your teaching accordingly.

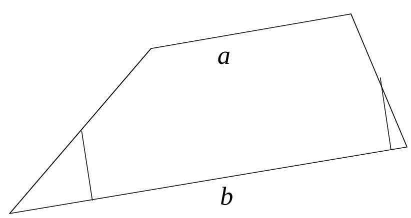
First, as a full class, work through the development of a formula for area of the parallelogram using one of the cutting algorithms that your class developed (use Problems 1–3 in the Student Module as a guide).

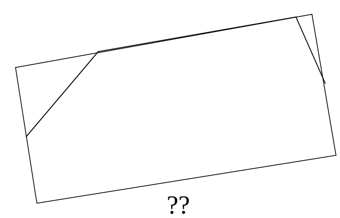
The class can then begin work on writing formulas for the triangle and trapezoid (Problems 4–8). Work in small groups should be effective, with each group presenting results at the completion of the investigation.

There are two remaining sections in the investigation: a dissection of the circle which shows one way to derive the circle area formula, and an extensive set of "Take It Further" problems. Use as many of these as time allows, working them in as further classwork and/or homework.

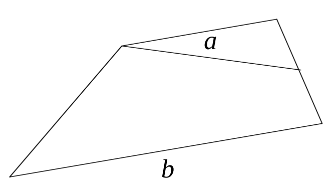
Since the dissections are not all identical, the formulas will not all be the same. Encourage students to demonstrate whether different formulas are algebraically equivalent.

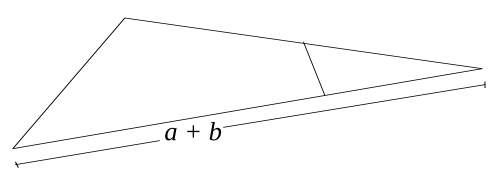
There are many trapezoid dissections, some of which make for easier area formulas than others. The following dissection, while elegant, makes the area formula difficult because we are not sure of the base of the rectangle.

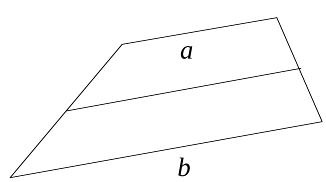


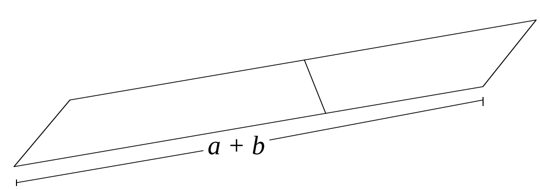


However, turning the trapezoid into either a triangle with base $a + b$ and height h or a parallelogram with base $a + b$ and height $\frac{1}{2}h$ will lead students more quickly to a formula:









This investigation contains a mathematically shaky definition of *altitude*, both for triangles and trapezoids. The fact that most triangles have three different altitudes makes wording a definition awkward, so the student materials are written to be more clear but less precise. You can decide if this is a problem for your class and develop a more precise definition if you wish.

ASSESSMENT AND HOMEWORK IDEAS.......

- After completing Problems 1–8, the "Checkpoint" problems would make suitable homework, or students could write up their work on the triangle and trapezoid, showing how their formula was developed.
- For an excellent assessment exercise, or to give students more practice in converting the dissection algorithms into formulas, give them a *new* figure to work with: Draw a picture of a kite. Show how the kite could be dissected into a rectangle. Then develop an area formula for the kite based on this dissection. Write up a report showing the dissection and how the formula relates to it.
- If the area formulas themselves are an important component in your curriculum, stop here and assign homework that requires using the formulas for calculation. Choose figures and problems from any standard text.
- The "Take It Further" section includes a set of problems about areas of regular polygons. Include problems as homework, with some class time for discussion.

USING TECHNOLOGY ....

Geometry software is helpful for Problem 24. Students should create a triangle and then find a way to move one vertex so that the area doesn't change. What shape does the vertex trace out?

MORE ON THE MIDLINE THEOREM

Materials:
- Cutting Kits

Technology: Geometry software is optional. See "Using Technology."

OVERVIEW

Cutting along midlines is one of the most popular techniques for solving the dissection problems in the previous investigation. Students quickly discover the simplicity of solutions that use these cuts. They have probably become familiar with the Midline Theorem, even if they haven't named it or proved it. The purpose of this investigation is to build a formal proof of the theorem. In doing so, students will have the opportunity to use the skills they have developed while studying congruence and proof.

Students may need to be reminded of the ways to prove a quadrilateral is a parallelogram.

Students should come into this investigation with some experience doing congruent triangle proofs, and with knowledge of the basic properties of parallelograms and parallel lines.

TEACHING THE INVESTIGATION

Alternate beginning: Assign the "Write and Reflect" exploration for homework the night before you plan to do this investigation. Begin class with students explaining how they convinced themselves that the Midline Conjecture is true.

Begin class with a review of the dissection pictured in the Student Module and a clarification of the student statement: "The midline is half as long as the base."

Students working in small groups should then explore midlines (as indicated in the "Write and Reflect" question) and share with the class how they investigated the problem. The class can then move to the development of the formal proof as outlined in the text. For many classes, this would best be done as a full class with the teacher leading the discussion, and the proof emerging on the chalkboard.

Have students post examples of other cutting algorithms that depended on the use of a midline.

If your class has already proved the Midline Theorem, omit the proof and move directly to the "Take It Further" exploration.

An interesting discussion should ensue if you ask: "The midline is half as long as the third side of the triangle? What if we created 'tri-lines' by connecting trisection points on the sides of a triangle? What is the relationship of the tri-line length to the length of the base of a triangle?"

ASSESSMENT AND HOMEWORK IDEAS.......

- Have students write out the full formal proof of the Midline Theorem for homework.
- Ask students to:
  1. find another dissection in which they used a midline cut;
  2. study this dissection;
  3. explain how each part of the Midline Theorem was important in making the dissection work. They would need to explain, for example, whether the fact that the midline is parallel to the base was necessary to make the dissection work.
- For an extra project or a journal writing exercise, students could organize and present all of their results from the "Take It Further" exploration of midlines in quadrilaterals. For a shorter homework assignment, they could explain and prove just one of their conjectures.

USING TECHNOLOGY ....

If your class has access to computers, allow the students to explore the triangle midline on the computer using geometry software. Use directions such as:

1. Draw a triangle. Draw a midline segment by connecting the midpoints of two sides of the triangle.
2. Measure the length of the midline, the length of the third side of the triangle, and all angles.
3. Stretch the triangle into all different sizes and shapes; study what happens to the measurements you made. Does the midline *always* measure half the base?
4. What other invariants can you find in this situation?

The "Take It Further" problem would make an even more exciting computer lab exploration. In fact, the investigation of midlines in quadrilaterals should be done on the computer, if at all possible. The constructions are not difficult; there are a lot of possible conjectures students can make; and their knowledge of triangle midlines should give them some intuition about what to look for in the quadrilateral situations.

Teaching Notes

Investigation **3.6**

Student Pages 53–70

The Pythagorean Theorem

Materials:

- **Cutting Kits**
- **graph paper**
- **transparency of proof**

Technology: Geometry software is optional. See "Using Technology."

Overview

Students explore and explain dissection proofs of the Pythagorean Theorem. Following this, a short historical essay describes the secret society of Pythagoreans. A varied problem set includes numerous examples involving calculation, searching for patterns, Pythagorean triples, and the distance formula.

To perform calculations using the Pythagorean Theorem, students will need to have some experience with square roots. The distance formula problems assume basic knowledge of the x–y coordinate plane, but these problems can be skipped if your students have no experience yet with coordinates.

The investigation centers around a dissection proof of the Pythagorean Theorem. Prepare an overhead transparency of this proof to facilitate class discussion and explanation. Study the problem set in advance to decide which topics are important and how much time you want to devote to each.

Teaching the Investigation

The night before this lesson begins, ask students to read the first part of the text where the Pythagorean Theorem is presented in historical perspective and also the reading about the Pythagorean Society.

The Pythagorean Theorem is certainly one of the major results in high school mathematics. We present it here because it is demonstrated so beautifully by dissection proof. One such proof is explained in detail in the text, and five more are presented later in the module in Investigation 3.12.

Using an overhead transparency which shows the shaded squares, ask a student to identify and label those segments which have lengths a, b, and c.

Focus the class activity on reading and explaining the dissection proof. Students can work with a partner or in small groups to read and understand the proof. When you think they have figured it out, allow one or more students to explain the proof to the whole class.

After the explanation, students should do the "Write and Reflect" Problem 3, reconstructing the entire proof from memory. If there is time, have students do Problems 4 and 5, which verify the dissection proof by calculation. The remaining problems can be completed as homework and in class the following day.

If you want to spend more time on proof or on the Pythagorean Theorem, or if your

class enjoys the challenge of figuring out dissection proofs, take time to tackle the remaining proofs that are found in Investigation 3.12.

Problems 12, 17, and 18 ask students to look for patterns. The patterns are only obvious if students express the lengths in radical form. If students are not familiar with simplified radicals, suggest that they look for a pattern in the squared sidelengths.

ASSESSMENT AND HOMEWORK IDEAS.......

- The Student Module problem set, readings, and "Take It Further" provide extensive homework possibilities.
- **Challenge** To maximize student experience with dissection proof, choose one of the additional proofs in Investigation 3.12; ask students to explain how it works.
- If this is your students' first exposure to the Pythagorean Theorem, you can supplement the Student Module with more practice problems.

USING TECHNOLOGY ....

If your geometry software comes with sample sketches demonstrating the Pythagorean Theorem, you may want to let students experiment with the sketches as part of the process of working through the proof. You can also visit the Math Forum at

http://forum.swarthmore.edu

for more sample sketches.

ADDITIONAL RESOURCES ....

The following sources were consulted in researching the "Perspective" essay on the Pythagorean Theorem and the Pythagorean Society. Students or teachers who want to learn more about these topics may wish to do some reading from these sources.

Calinger, Ronald. *Vita Mathematica: Historical Research and Integration with Teaching*. (MAA Notes) The Mathematical Association of America, 1997.

Eves, Howard. *An Introduction to the History of Mathematics*, 4th ed. Holt, Rinehart & Winston, 1990.

Kline, Morris. *Mathematics in Western Culture*. Oxford University Press, 1964.

"Past Present (we)—Present Future (you)." Association for Women in Mathematics Newsletter 9(6), Nov/Dec 1979, 11–17.

Russell, Bertrand. *History of Western Philosophy*. Simon and Schuster, 1975.

MATHEMATICS CONNECTIONS ....

The search for *Pythagorean triples* (integers which can be lengths of the three sides of a right triangle) leads to surprising applications of both the algebra of the Gaussian integers and the algebra of points on the plane with rational coordinates.

But this is just the beginning. The general question "Can you find a geometric figure that has some collection of specified parts whose measures are in a particular algebraic system?" leads to some fascinating algebraic questions. This "Diophantine geometry" is concerned with questions like these:

- Can you find points A, B, and C on the plane with integer coordinates so that $\triangle ABC$ has integer sidelengths?
- Which integers are areas of right triangles whose sidelengths are rational numbers?
- Are there any scalene triangles with integer sidelengths and a 60° angle?
- Are there any scalene triangles with integer sidelengths and integer area?

What if you insist that the triangle has no horizontal or vertical sides and is not a right triangle?

Are there any scalene triangles with rational sidelengths and rational area?

These questions all sound alike, but, as is typical in algebra, some are quite simple to solve and some are amazingly difficult. (Question 3.6 remains an open problem, although significant progress has been made in the last two decades.)

Not surprisingly, there are some coherent methods from algebra that can be used to investigate problems like these; in this section, we'll look at two such methods, "norm equations" and "secants and conics."

Finding Pythagorean triples amounts to finding triples of integers (a, b, c) so that $a^2 + b^2 = c^2$. If you are "thinking Gaussian integers," the form $a^2 + b^2$ should look familiar. It is the *norm* of the Gaussian integer $a + bi$. Just to refresh your memory, here are the relevant definitions and properties:

. . . but were afraid to ask.

All you ever wanted to know about conjugation and Norm

1. If $z = a + bi$ is a Gaussian integer, the "complex conjugate" of z, written $\overline{z}$, is defined by $\overline{z} = a - bi$.

2. Using this definition, the following properties of conjugation hold:

   a. $\overline{z + w} = \overline{z} + \overline{w}$ for all Gaussian integers z and w.

   b. $\overline{zw} = \overline{z}\,\overline{w}$ for all Gaussian integers z and w.

   c. $z = \overline{z} \Leftrightarrow z \in \mathbb{R}$

   d. $z\,\overline{z} = a^2 + b^2$, a nonnegative integer.

3. The norm of z, written $N(z)$, is defined as the product of z and its complex conjugate: $N(z) = z\,\overline{z}$.

4. Using this definition, the following properties of norm hold:

   a. $N(zw) = N(z)\,N(w)$ for all Gaussian integers z and w.

   b. $N(z) = a^2 + b^2$, a nonnegative integer.

Exercise: Show that, if z is a Gaussian integer, then

$$N(z^2) = (N(z))^2 .$$

Notice that the right side of this equation is a *perfect square* (it is the square of an integer).

This exercise is a key to one of the nicest ways around for generating Pythagorean triples. The idea goes like this:

- The equation $a^2 + b^2 = c^2$ can be written $N(z) = c^2$ where $z = a + bi$. So, we are looking for Gaussian integers whose norms are perfect squares.
- The exercise above says that the norm of a Gaussian integer will be a perfect square if the Gaussian integer is itself a perfect square.
- So, to generate Pythagorean triples, pick a Gaussian integer at random, and square it. The square will be a Gaussian integer $a + bi$ whose norm, $a^2 + b^2$ will be a perfect square. That is, $a^2 + b^2$ will equal c^2 for some integer c, and (a, b, c) will be a Pythagorean triple.

The Pythagorean Theorem (continued)

Can you:

1. State this method precisely and prove that it works?

2. Implement this method in your computer algebra system, and use it to generate a few hundred Pythagorean triples?

3. This method produces duplicates and sometimes produces negative "legs." Refine the algorithm so that it produces only positive triples and produces no duplicates.

4. Even after you eliminate duplicates, there are annoying triples like (6, 8, 10) that show up and are simple multiples of a "primitive" triple (this one is twice (3, 4, 5)). Characterize those z so that z^2 will generate a *primitive* Pythagorean triple.

Well, there are equilateral triangles, but how about scalene ones?

The problem of finding Pythagorean triples asks for integer-sided triangles with a right angle. A natural generalization is to ask for integer-sided triangles with some *other* kind of angle. For example, are there any triangles with integer sidelengths and a 60° angle?

Suppose there were.

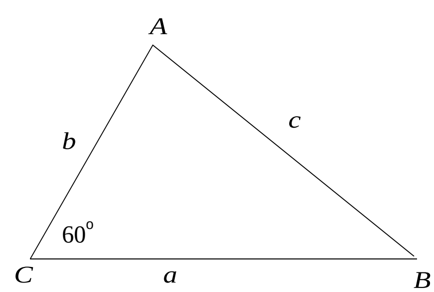


Figure 1: Integer sidelengths, $m\angle C = 60^\circ$

In the case of a right triangle, the Pythagorean Theorem gave us a relationship among the three sides ($a^2 + b^2 = c^2$). In a triangle where $\angle C$ is not a right angle, $a^2 + b^2$ is *not* the same as c^2, but there is a theorem that generalizes Pythagoras and tells us how the sides are related:

THEOREM *The Law of Cosines*

If the sides of a triangle are labeled as in Figure 2,

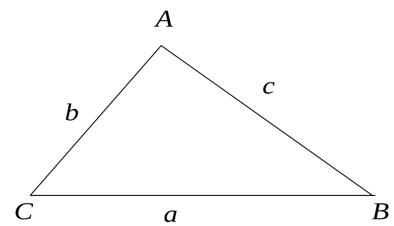


Figure 2

then $c^2 = a^2 + b^2 - 2ab\cos C$.

Here, "C" means the measure of ∠*BCA*.

So, let's go back to Figure 1:

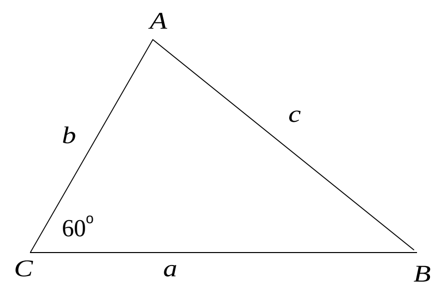


Figure 1: Integer sidelengths, $m\angle C = 60^\circ$

By the Law of Cosines,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos 60^\circ \\ &= a^2 + b^2 - 2ab \cdot \frac{1}{2} \\ &= a^2 + b^2 - ab. \end{aligned}$$

So, finding the kind of triangles we want amounts to finding triples of integers (a, b, c)

so that $a^2 - ab + b^2 = c^2$. Just as before, we are looking for $a, b \in \mathbb{Z}$ so that $a^2 - ab + b^2$ is a perfect square. Some wishful thinking is in order:

> If $a^2 - ab + b^2$ were the "norm" of something, and if that norm function behaved like the ordinary norm from the Gaussian integers (in particular, if the norm of a product were the product of the norms), then we'd be able to use the same method: Take a thing, square it, and then it's norm would
>
> - have the right form ($a^2 - ab + b^2$);
> - be a perfect square.

But, you say, this is only wishful thinking. The norm function is what it is. If $z = a + bi$, then $N(z) = a^2 + b^2$, *not* $a^2 - ab + b^2$. Norm is norm, and you can't change it.

Well, it's not the norm of $a + bi$, but suppose it were the norm of $a + b\omega$ for some complex number ω. Let's work backwards and see if we could figure out what ω would have to be. Remember, the norm is the product of the number and its conjugate, so, if a and b are integers,

$$\begin{aligned} N(a + b\omega) &= (a + b\omega)\left(\overline{a + b\omega}\right) \\ &= (a + b\omega)\left(\overline{a} + \overline{b\omega}\right) \\ &= (a + b\omega)\left(\overline{a} + \overline{b}\,\overline{\omega}\right) \\ &= (a + b\omega)\left(a + b\,\overline{\omega}\right) \\ &= a^2 + ab(\omega + \overline{\omega}) + b^2(\omega\,\overline{\omega}), \end{aligned}$$

As usual, justify each step in the above calculation.

and if we want this to be $a^2 - ab + b^2$, then we want

$$\begin{aligned} \omega + \overline{\omega} &= -1 \quad \text{and} \\ \omega\,\overline{\omega} &= 1. \end{aligned}$$

Well, that pretty much nails ω down: we know the sum of ω and its complex conjugate (it's -1), and we know the product $\omega\,\overline{\omega}$ (it's 1). So, ω is a root of the quadratic equation

$$x^2 + x + 1 = 0.$$

This is because

$$x^2 - (\text{the sum of the roots})\, x + (\text{the product of the roots}) = 0.$$

The Pythagorean Theorem (continued)

Using the quadratic formula, we can take ω to be

$$\omega = \frac{-1 + i\sqrt{3}}{2},$$

The other root is then $\overline{\omega} = \frac{-1-i\sqrt{3}}{2}$. That will work, too.

and we can now generate as many triples of integers (a, b, c) so that $c^2 = a^2 - ab + b^2$ as we like.

Example: Start with $z = 3 + 2\omega$. Square it:

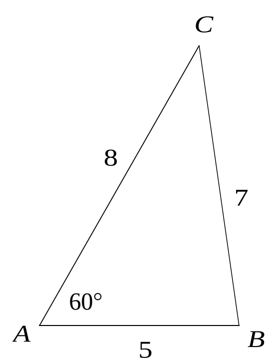
$$\begin{aligned} z^2 &= (3 + 2\omega)^2 \\ &= 9 + 12\omega + 4\omega^2 \\ &= 9 + 12\omega + 4(-1 - \omega) \quad \text{Don't forget: } \omega^2 + \omega + 1 = 0 \\ &= 5 + 8\omega \end{aligned}$$

and voilà:

$$5^2 - 5 \cdot 8 + 8^2 = 49 \quad \text{a perfect square!}$$

Or, a triangle with sides of length 5 and 8 and an included angle of 60° has a third side of length 7.

So the triangle whose sides have length 5, 8, and 7 has a 60° angle.



A (5, 8, 7) *triangle has a 60° angle.*

THE PYTHAGOREAN THEOREM (continued)

Notice that, when we were looking for a triangle with integer sidelengths and a 60° angle, we were led to search for a complex number ω so that

$$\omega + \overline{\omega} = -1 \quad \text{and}$$
$$\omega\overline{\omega} = 1.$$

This meant that $\omega^2+\omega+1=0$, producing two choices for ω. In fact, what we really needed about ω was this defining behavior ($\omega^2+\omega+1=0$) and *not* it's precise value ($\frac{-1+i\sqrt{3}}{2}$). We just needed to know that, when you calculate with "expressions" in ω, powers higher than 2 can be "folded back" using the equation $\omega^2=-\omega-1$. This idea of using the behavior of something instead of its value is an important one in algebra. It's also very close to the way many high school students deal with complex numbers: they treat "i" as a formal symbol whose square can be replaced by -1 anytime it shows up. Many teachers think this is mindless mechanical behavior; maybe we're too hard on students.

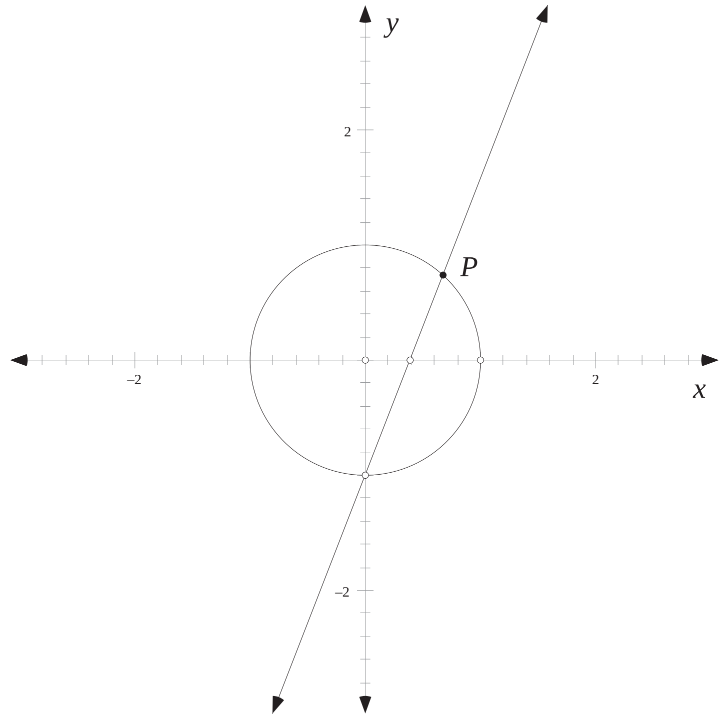
"Good mathematics is not how many answers you know, but how you behave when you don't know the answer." – anon.

Exercise: The problem of finding a triangle with integer sides and one angle θ comes down to finding a complex number α so that the norm of $a+b\alpha$ has the form that comes from applying the law of cosines to a generic triangle of the desired type. Express α in terms of θ, and give a condition that θ has to meet in order for this problem to have a solution.

A Geometric Approach

There is another way to generate Pythagorean triples, using the unit circle and coordinate geometry:

The Pythagorean Theorem (continued)



$P = (\frac{a}{d}, \frac{b}{e})$

The equation of the unit circle is $x^2 + y^2 = 1$. So, if we can find "rational points" on the circle, say $(\frac{a}{d}, \frac{b}{e})$, then we have

$$\begin{aligned}\left(\frac{a}{d}\right)^2 + \left(\frac{b}{e}\right)^2 &= 1\\ a^2e^2 + b^2d^2 &= d^2e^2\\ (ae)^2 + (bd)^2 &= (de)^2,\end{aligned}$$

and the numbers ae, bd, de are a Pythagorean triple. But how do we find these points?

Theorem

If a line passes through the point $(0, -1)$ and has rational slope, then its other intersection with the unit circle will be a rational point.

The Pythagorean Theorem (continued)

Proof: The equation of the line is $y = \frac{a}{b}x - 1$, where a and b are integers. Substitute for y in the equation of the circle to get

$$\begin{aligned}
x^2 + \left(\frac{a}{b}x - 1\right)^2 &= 1 \\
x^2 + \left(\frac{a^2}{b^2}\right)x^2 - \frac{2a}{b}x + 1 &= 1 \\
\left(\frac{a^2+b^2}{b^2}\right)x^2 - \frac{2a}{b}x &= 0 \\
(a^2+b^2)x^2 - 2abx &= 0 \\
x &= \frac{2ab}{a^2+b^2}.
\end{aligned}$$

Since a and b are integers, $\frac{2ab}{a^2+b^2}$ is a rational number, so x is rational. A similar argument shows that y is rational as well.

Now look back at the problem of finding triangles with integer sides and one 60° angle. You want triples where $a^2 - ab + b^2 = c^2$. Could you use a similar geometric approach to generate them?

- Graph $x^2 - xy + y^2 = 1$. What kind of object is it?
- If you have a rational point on the graph of $x^2 - xy + y^2 = 1$, can you use it to find the integer-sided triangle you are looking for?

Theorem

If a line passes through the point $(0, -1)$ and has rational slope, then its other intersection with the graph of $x^2 - xy + y^2 = 1$ will be a rational point.

Exercise: Can you prove this theorem?

Teaching Notes

Investigation **3.7**

Student Pages 71–79

CHANGING SHAPE

OVERVIEW

Materials:
- **Cutting Kits**

More dissection problems build on the students' skill and understanding of dissection, area, and congruence. Algorithms are developed for cutting problems that have new kinds of restrictions. A triangle is dissected into a new triangle with restrictions placed on the number of cuts, the type of cuts, or the shape of the final triangle. Another problem asks students to dissect a rectangle into a same-area rectangle that has a given length base. One algorithm for this cutting problem is presented in the text and analyzed in some detail by students. Many of the problems will be solved by composing two or more previously established cutting algorithms (a problem-solving strategy that is new to many students and quite exciting to discover).

Investigations 3.2 and 3.3 are prerequisites. Some problems are difficult, others are less difficult. You may want to solve the problems yourself or look through the Solution Resource before deciding which problems to assign.

TEACHING THE INVESTIGATION

There are three main parts to this investigation:

- Dissection problems with triangles (Problems 1–4);
- The rectangle algorithm analysis (Problems 5–11);
- Dividing shapes into a given number of equal area parts (Problems 12–17).

To speed up the Investigation, give Problems 1 and 2 to half the groups, and Problems 3 and 4 to the other half. Classwork time will be shortened, but there will still be a chance to see more than one solution for each problem.

Each part will work well if taught as a group activity followed by student presentations and a discussion of the validity of different solution methods. At this point in the module, students should be fairly adept both at solving the dissection problems and at explaining/critiquing the solutions.

Students will almost certainly begin to compose solutions from several previously proven algorithms. It would simplify the explanations and validate student work if some of these algorithms were given names. Augustus could explain by saying: I made my triangle into a parallelogram by June's method. Then I made this parallelogram into a new triangle using Julio's method.

Problem 12 (the division of a triangle into triangles of equal area) presents a good opportunity for discussing and reviewing the triangle median, midline, altitude, and angle bisector. As students make their explanations, encourage correct use of terminology and ask students to prove some of the results.

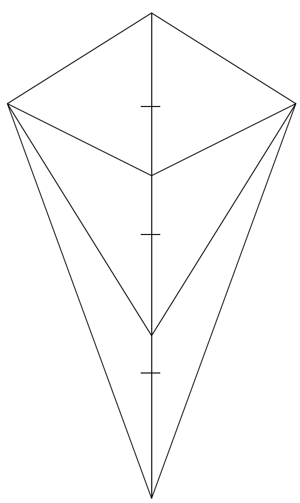
ASSESSMENT AND HOMEWORK IDEAS.......

For homework ask students to:

- Write up the solution to one of the dissection problems with an explanation of why it works.
- If Problems 1 and 3 were completed in class during group work, assign Problems 2 and 4 for follow-up homework.
- After the Rectangle Division algorithm is analyzed in class, assign the "Write and Reflect" problem as homework.
- Problem 18 would make a good assignment after the students have finished Problem 12 (division into equal parts).

Informal assessment will be ongoing in this investigation as student presentations and homework demonstrate their increasing understanding of this material. For more formal assessment:

- Grade the presentations of solutions.
- Assign "Take It Further" Problem 20 as a journal entry that you will read and grade.
- Assign "Take It Further" Problem 20 as a group task, requiring each person in the group to write up one part. The completed parts a through d would be turned in together and graded for group and individual credit.
- Give a quiz. Give students a figure of a kite, with the main diagonal trisected as shown.



Ask students to explain whether this cuts the kite into six triangles of equal area. Would this method work to divide any other type of quadrilateral (for example, a rhombus, trapezoid, or scalene quadrilateral) into equal area triangles?

USING TECHNOLOGY

A computer script or macro could be written by students to demonstrate the rectangle-to-rectangle algorithm. Once written, it would serve the class well for demonstrations and for discussing the limitations of the faulty algorithm given in the Student Module.

Problem 18 in the "Checkpoint" and Problem 20 in the "Take It Further" section could both be done on the computer. The directions printed in the Student Module are still suitable. Students will simply use geometry software to experiment, to verify which statements *appear* to be true, and to construct counterexamples for the false statements. If these problems are done in the computer lab, you might want students to present their solutions on the computer rather than in writing.

EQUIDECOMPOSABLE FIGURES

Theorem: If two rectilinear figures have the same area, then they are equidecomposable.

Materials:
- **Cutting Kits**

OVERVIEW

A proof of the Bolyai-Gerwien Theorem is presented, including four lemmas that are necessary to the proof. Students are led through the lemmas and to the proof by a series of small steps, each of which makes sense because it relates so closely to the work students have been doing in this module.

Read the Bolyai-Gerwien Theorem proof ahead of time to estimate the level of difficulty it poses for your students and the time it will take them to understand it. Plan homework and class time accordingly.

Certain results that have been established in this module are important in understanding the proof in this investigation:

- If one figure can be dissected and rearranged into another figure, then the two figures have equal area.
- Every triangle can be dissected into a rectangle.
- It is possible to dissect a rectangle into a new rectangle with a given length base (see Investigation 3.7).

TEACHING THE INVESTIGATION

The night before you start the investigation, students should read the first page of the investigation, and complete the "Write and Reflect" problem. (You might also want them to write definitions of lemma, converse, rectilinear, and equidecomposable.)

It might not be obvious to many students what the Bolyai-Gerwien Theorem is saying, and even less clear why one might need to prove it. A discussion of the cases presented in the "Write and Reflect" problem should convince them of the need for the proof, and vocabulary can be clarified if necessary.

An alternate approach: Assign each group the job of justifying *one* of the lemmas. Follow these presentations with a discussion of the full proof.

The proof: You may want to simply claim the pleasure of leading your class through the lemmas and the proof. Or, you might decide to have students work through the text explication and problems by reading, discussing, and writing in their small working groups. It is written in a style that would support this sort of self-taught approach. Whatever you choose, conclude with a full class run-through of the complete proof, encouraging students to participate in the final explanation.

ASSESSMENT AND HOMEWORK IDEAS.......

- If the class does not complete the "Write and Reflect" problems in class, students should certainly do them for homework.
- The reading about Professor Chih-Han Sah could also be used as a homework assignment.
- The demonstration: When the proof is complete, each student (or group) should work through an illustrated version of the steps for a particular pair of figures (see "Write and Reflect" Problems 8–10). Presentation of these can serve as an assessment for the investigation.

Teaching Notes

Investigation **3.9**

Student Pages 87–93

AREA AND PERIMETER

Materials:

- **Cutting Kits**
- **construction tools**
- **a lot of tape**
- **3 × 5 cards**
- **colored marking pens**
- **graph paper**

Technology: Geometry software is optional. See "Using Technology."

If no special drawing tools are available, students can mark off two sides of a 3 × 5 card in $\frac{1}{8}$ or $\frac{1}{4}$ inch units. If the card is placed correctly, tracing along its edges will give an almost perfect isosceles right triangle.

OVERVIEW

In this investigation, students study what happens to the perimeter of a figure under dissection. Studying some of their previous cutting algorithms, they establish first that perimeter does *not* remain invariant. Then, working with an iterative cutting algorithm on a square, they investigate whether the perimeter ever changes in a predictable manner. If the dissection of the figure follows a repeated pattern, is there a pattern to the increase in the perimeter?

In preparation, students must have accomplished the basic dissections of parallelogram, triangle, and trapezoid to rectangle. (These were done in Investigation 3.2.) The major decision you must make in preparation is whether you wish to do the investigation on computers.

In the square dissection (Problems 4–9) students must repeatedly construct isosceles right triangles from a given hypotenuse. They will need some kind of construction tool to assist them in this.

TEACHING THE INVESTIGATION

Students should be *completely* convinced by now that area is invariant under dissection. But what about perimeter? Pose this question, and put them right to work on Problems 1 and 2. Group work would be advantageous here, because each student could dissect a different-shape parallelogram or use a different algorithm for dissection, and hence the group would end up with more evidence to use in drawing conclusions.

Complete the lesson with the "Write and Reflect" Problem 3. Students will quickly conclude that the perimeter is not invariant, but may need to experiment and think more to be able to describe *how* it changes. This problem can be done in groups, followed by a group presentations, or used as homework. In either case, ask students to explain why the perimeter changes the way it does.

AREA AND PERIMETER (continued)

Warm-up problem: Start with an 8 × 8 square (perimeter = 32). Cut it in half vertically and horizontally. Now you have four 4 × 4 squares. Line them up end to end to make a long rectangle (4 × 16). Cut twice again and line these rectangles up end to end to make an even longer rectangle (2 × 32) . . . cut and line up again and again. What do we know about these rectangles? About their areas? Will there be a predictable pattern for the perimeters?

On the second day, introduce the square activity with the question: "Imagine a dissection that gets repeated many times. Do you think we could predict a pattern for the repeated changes in perimeter?" Try the warm-up problem to get students started, then move to the square dissection (Problems 4–10). The rules for dissecting the square are clearly outlined in the Student Module, so, again, set them right to work to see what they can come up with.

In Problem 6 and again in Problem 9, after students have completed the dissection algorithm, they are asked to determine the perimeter of the new figure. Measurement is possible, but tedious and inaccurate. Encourage them to compute the perimeter by studying the way the figure was formed. (They can use the Pythagorean Theorem or the ratio of side to hypotenuse in a 45-45-90 right triangle, if they know it.) Analysis of the emerging pattern will also be much easier using these calculations rather than measurement.

When the activity is complete, students should write a report summarizing their conclusions or make presentations to the class.

The square dissection can be done by drawing instead of cutting. Just start with a 16 × 16 (or 32 × 32) square drawn on *big* graph paper. For each full application of the algorithm, sketch the changes in pencil, then draw the completed new form with a different color pen. With graph paper, the isosceles triangles are easy to draw.

The square problem opens up the whole topic of recursion and fractals. Adding in a few more activities on those topics will delight many students.

The figure that is formed after one full application of the square dissection algorithm *will* tile the plane. (Algorithms like this can be used to make tessellation drawings.) Ask students if the second or third stage figures also tile the plane.

ASSESSMENT AND HOMEWORK IDEAS.......

- Some classes have used the square problem as a project that students completed at home, turning in their final drawings and results in colorful poster form.
- If the 45-45-90 triangle pattern of sidelengths is new to your students, add some homework here that will reinforce it.
- Have students predict the perimeter of the square fractal after 10 repetitions of the cutting algorithm.

USING TECHNOLOGY

Problems 1 and 2: These problems are *very* suitable for computer investigation. Using geometry software, students can easily study how the perimeters compare, see what limits there are on the ratio, and conjecture about what factors affect the ratio.

If students have previously constructed dissections in the lab, you won't have to allow much time for that part of the problem. They can move directly to studying the perimeters. If they haven't, allow time for them to come up with the constructions. Or you can give them construction directions for one algorithm, allowing most of the class time to be spent investigating the perimeters.

For most geometry software, the parallelogram-to-rectangle dissection will work with directions like these:

1. Construct an angle $\angle ABC$ that can change in size as you drag on any of its points.

2. Construct segment $\overline{AD}$ parallel to $\overline{BC}$ and segment $\overline{CD}$ parallel to $\overline{AB}$, completing parallelogram $ABCD$.

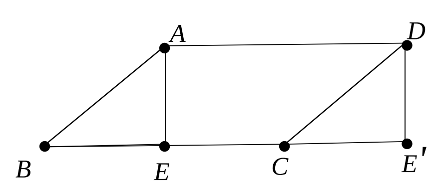
3. Measure the perimeter of $ABCD$.

4. By dragging a corner, adjust the shape of your parallelogram to make $\angle A$ obtuse.

5. Construct $\overline{AE}$ perpendicular to $\overline{BC}$.

6. Translate $\triangle ABE$ along $\overline{BC}$ so that B is mapped to C and A is mapped to D. The resulting figure should show a rectangle $AEE'D$.

7. Measure the perimeter of $AEE'D$.



Perimeter $ABCD$ = 13.19 cm
Perimeter $AEE'D$ = 11.25 cm
$\dfrac{\text{Perimeter } ABCD}{\text{Perimeter } AEE'D} = 1.17$

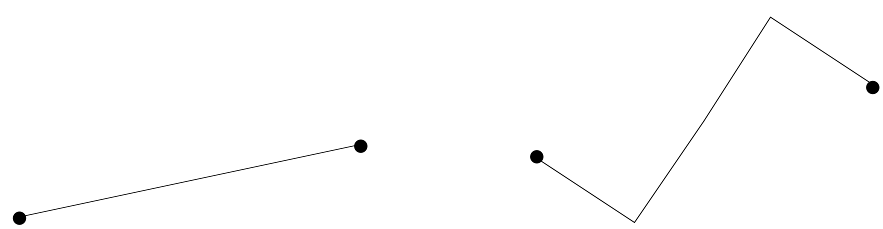
8. Compare the two perimeters as you drag the parallelogram into different shapes.

9. What is the relationship of the perimeters? Is it an invariant relationship? Are there limits to the ratio of the parallelogram perimeter to the rectangle perimeter? Why?

10. Write a report, including sketches and calculations, to summarize your investigation. State all conclusions and conjectures and justify them with evidence and argument.

The square dissection can also be done in the computer lab. Using geometry software, students could write a script or macro to perform the cut-and-paste part of the square dissection, and then easily study the changing perimeter with the software's measurement tools. Warning: Creating the macro would be very challenging and only realistic as an activity for students who are very familiar with the software and with writing macros.

You could provide students with such a script. Given a segment and two endpoints as input (so the computer knows which direction to rotate), it should create the cutout:



Teaching Notes

Investigation **3.10**

Student Pages 94–96

MAKING THE MOST OF PERIMETER

OVERVIEW

The problem: In building a rectangular house with 128-foot perimeter, what dimensions should you choose to give maximum area?

Materials:

- **Cutting Kits**
- **Overhead transparencies**
- **Graph paper**

Technology: Geometry software is optional. See "Using Technology."

A single maximization problem is presented for students to solve, and a dissection proof is shown for them to read, study, critique, and generalize. In doing this investigation, students will increase their understanding of the relationship between area and perimeter in polygons, and study dissection proof in a context that should make sense to them.

To facilitate class discussion, prepare an overhead transparency of the dissection proof in the Student Module. If students will be presenting their own proofs to the class, be sure to have blank transparencies or newsprint available. This investigation builds on experiences from the entire module, but does not require them for success. It could even work as a stand-alone activity.

TEACHING THE INVESTIGATION

The importance of this investigation is in its treatment of proof. Construct the lesson so that students have the opportunity to do these things:

- Solve the maximization problem themselves, so that they have a good intuitive feel for what the problem is about.
- Study the proof in the Student Module and ask: What does it purport to prove? Does it succeed? Why?
- Repeat the proof on a different rectangle.
- Write a generalized form of the proof for an $a \times b$ rectangle.

Put the problem on a separate piece of paper for this part of the class so that students won't be looking at the proof while they are trying to explain it themselves.

Assign students Problem 2 to solve by working together in groups. Explain that they not only have to solve the problem, but also have to present an explanation of why their solution is the best. When these presentations are complete, introduce the proof that is shown in the Student Module. Focus class discussion on the viability of this proof.

For homework, assign the remaining problems, in which students replicate and then generalize the proof. Use their written homework for assessment purposes.

An alternate plan: For some classes, Problem 2 itself could be done as preview homework. Classwork would then focus on discussing the proof in the Student Module and trying it out on other rectangles. For follow-up homework, students would then rewrite the proof and work on generalizing it for an $a \times b$ rectangle.

ASSESSMENT AND HOMEWORK IDEAS.......

Considering the placement of this investigation near the end of the module, assigning the full set of "Write and Reflect" questions as a take-home assessment would be a viable plan. See "Teaching the Investigation" for additional suggestions.

USING TECHNOLOGY ....

The primary emphasis in this investigation will be on analyzing the dissection proof. If, however, your class has the time and interest in investigating the maximization problem itself, then geometry software provides an ideal tool. Here is a suggested outline:

1. Construct a rectangle which can be stretched to have varying lengths and widths, but has a constant total perimeter.

2. Measure the length and width, and compute the area of the rectangle.

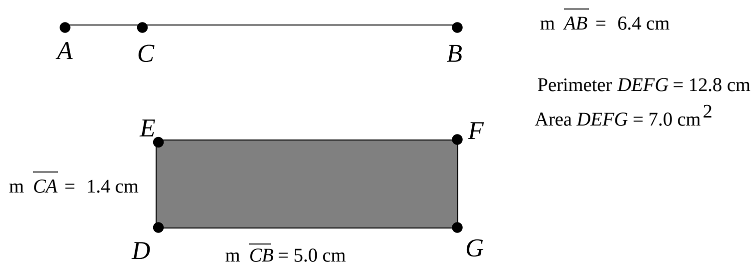
3. The perimeter is always the same, but how does the area change as the length and width change? What is the minimum possible area? What is the maximum possible area?

4. Write an explanation of what you have found about the relationship between area and the dimensions of the rectangle using charts, graphs, or other evidence from your computer exploration to clarify and justify your explanation.

Constructing the rectangle with constant perimeter may be difficult for some students, especially if they haven't had a lot of experience with the software. If this is the case, you could substitute more detailed instructions such as these:

MAKING THE MOST OF PERIMETER (continued)

1. Construct $\overline{AB}$ with point C on it. (Make sure C is moveable.)
2. Construct a pair of perpendicular lines and mark their intersection as point D.
3. Construct a circle with center D and radius $\overline{AC}$. Mark its intersection with one of the lines as point E.
4. Construct another circle with center D and radius BC. Mark where it intersects the other perpendicular line as point G.
5. From E, construct a line perpendicular to $\overline{DE}$.
6. From G, construct a line perpendicular to $\overline{DG}$.
7. Where these two lines meet, label the point F. This should be the fourth corner of a rectangle.
8. Construct segments for the sides of the rectangle and hide all construction circles and lines (but do *not* hide $\overline{AB}$).
9. Measure the perimeter and area of $DEFG$.
10. Now, move point C along the $\overline{AB}$. As C moves, the rectangle should change in shape, but keep a constant perimeter.

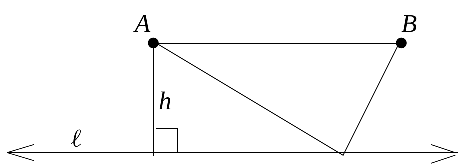


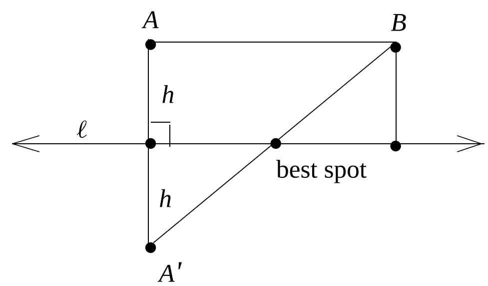
Whether students figure out their own methods of constructing the constant perimeter rectangle or use the method described above, the changing area will be immediately apparent. Students may even be intrigued enough to try to determine a function or pattern in the relationship between the sidelengths and the area.

MATHEMATICS CONNECTIONS . . . .

Here is another way to look at Problem 7:

If you are familiar with the "burning tent" problem from the module *Optimization: A Geometric Approach*, then you can think of it this way: You have two points A and B and a line ℓ parallel to $\overline{AB}$ that is distance h from $\overline{AB}$ (where h is the given height of the triangle and AB is the length of the base). You want to minimize the "trip" from A to ℓ to B.





One way to find the point on ℓ that minimizes the trip is to reflect A over ℓ, creating a new point A'. For any point P on ℓ, the distances AP and $A'P$ are equal. So you can create segment $A'B$, and the point where this crosses ℓ is the point which minimizes the trip. Because A and B are equidistant from ℓ (remember that ℓ is parallel to $\overline{AB}$), $\overline{A'B}$ will cross ℓ at the midpoint of the projection of $\overline{AB}$ onto ℓ. In other words, it will be at the point that creates an isosceles triangle.

Analyzing Dissections

Overview

If all has been going well, a great deal of assessment has occurred during the regular class activities and homework of this section of the module. Many teachers may still wish to complete this section with a final assessment activity to encourage individual or group accountability, to reinforce main concepts, to test mastery, or simply to provide a closing grade. The problems in this section are provided to use for any of these purposes. The problems are described in terms of possible uses, level of difficulty, and grading.

Teaching the Investigation

Dialogue Problems Two problems are included that are written in the form of short dialogues. The questions that follow the dialogue call for students to analyze the suggested solution to a dissection problem. Both problems are fairly easy and assess a student's understanding of the basic dissection problems. They could be used for quizzing early in the module (after Investigations 3.2 and 3.3). If the majority of the work in this module occurs in groups, using these dialogues for a short individual assessment would help supply information about individual learning, and would encourage individual accountability.

Supplementary Problems These 11 problems are suitable for individual or group end-of-unit assessment. Many uses are possible.

You can assign problems to the groups in a random or calculated manner, choosing problems that push each group to its limit.

- Assign each class group one problem to do and present to the class. Have each group do a different problem. Their group presentation will be graded as a demonstration of mastery of the material and of skill in group work.
- Assign one or two problems as an individual take-home assignment, to be written up and turned in for grading.
- Have an in-class individual test. Choose one or two problems; have students write a solution and explanation. Supplement this with short answer questions on vocabulary, properties of polygons, the Pythagorean Theorem, and so on.

ANALYZING DISSECTIONS (continued)

Level of difficulty:

Problem 11: Moderate/easy; the main methods used in class to cut a parallelogram into a rectangle probably did not make use of a diagonal cut. Students will probably cut along the diagonal immediately, then form two rectangles from the two triangles and join them.

Problem 12: Moderately difficult; requires several stages, subtracting areas of $\triangle APM$ and $\triangle BPM$ from the area of the larger triangle.

Problem 13: Easy; students can use any of the main triangle-rectangle methods they have already done in class.

Problem 14: Moderately easy; similar to Problem 11

Problem 15: Moderate; requires several stages, but is not tricky

Problem 16: This is easy once students think this through. If a wrong start is made, it becomes difficult.

Problem 17: Moderate; most likely solution involves subtracting equal areas as in Problems 12 and 15. This problem is more difficult if students are not familiar with the properties of lines drawn parallel to the side of a triangle.

Problem 18: Difficult

Problem 19: Moderately easy; subtraction of the overlapping rectangle produces a solution.

Problem 20: If the class has developed a single-cut algorithm for turning a parallelogram into a rectangle and a single-cut algorithm for turning a triangle into a parallelogram, then this will be moderately easy. Otherwise, it is difficult. Two stages are required in any case.

Problem 21: Challenging

Pythagorean Cutting Proofs

This section of the module, Investigations 3.12–3.15, contains extensions and enrichment activities that relate to cutting, dissections, and area. Use the activities anytime during the unit on cutting, at the end of the module for a culminating activity, or later in the year as enrichment and extensions.

Materials:
- Cutting Kits
- poster board, transparencies, and/or newsprint for student presentations

Technology: Geometry software is optional. See "Using Technology."

Overview

Investigation 3.6 contains one picture proof of the Pythagorean Theorem. Here five additional proofs are shown. Students can work individually or in groups to try to understand one of the proofs and then explain it to the rest of the class.

Students should study the picture proof in Investigation 3.6 before trying these proofs. The proofs are quite challenging. It might be advisable to give students the problems a few days in advance so that they have time to ponder them before trying to discuss them in groups or present them to the class.

Teaching the Investigation

The delight of teaching this investigation is in sharing with students the simplicity and elegance of this form of proof. Their entire notion of proof may be changed.

For a two-day activity, students could work together in groups, with each group solving one puzzle. They would then present their solutions the second day. If more time is available, the entire set of proofs could be presented to the class, and students could work in their groups to solve as many of the proofs as possible. As each solution emerges, a group or individual could be chosen to present it. The full class would then be able to participate in discussing or refining the solution. Following this method, the most difficult proofs would naturally be left for last, and could be assigned as ongoing homework until finally all were complete.

Assessment and Homework Ideas

Having students solve and present one of these proofs would work well for assessment purposes. Evaluate students on clarity of explanation, understanding of mathematics, and effectiveness of visual aids used in explanation.

USING TECHNOLOGY

The "Take It Further" problem "What if They Aren't Squares?" is quite successful as a computer exercise. The constructions are not too difficult if students have some previous experience with geometry software. Sample directions follow:

In the Student Module, there is a picture proof of the Pythagorean Theorem showing a right triangle with a square constructed on each side. Have you ever wondered whether those absolutely *had* to be squares? If they were semicircles or some other polygon, would the areas of the two smaller ones still add up to the area of the biggest one? Use a computer to investigate.

1. Construct a right triangle using geometry software.

You may have to leave the whole circle showing in the sketch. Some software won't hide half the circle.

2. On each side of the right triangle construct a semicircle. Make sure that the diameter of the semicircle is the same as the length of the side of the right triangle.

3. Now calculate the areas of the three semicircles. What is their relationship?

4. Investigate what happens if you construct equilateral triangles on all three sides of the right triangle, or rectangles, or polygons with "weird" shapes.

Be sure to explain any rules that determine which shapes work.

5. Can the Pythagorean Theorem be "Unsquared"? Prepare a written summary of what you have found, with drawings to illustrate your work.

6. For an extra challenge, include an algebraic explanation of your findings.

A Cutting Paradox

Materials:
- Cutting Kits
- graph paper
- calculators (for "Take It Further")

Overview

In this investigation, students cut up one rectangle and rearrange the pieces to form a different rectangle, discovering that the two rectangles have *different* areas. In analyzing this cutting paradox, they discover the need for rigorous analysis in such dissection problems and also have the opportunity to learn something about Fibonacci numbers. The investigation can be done as a simple stand-alone exploration or can fit neatly into the sequence of cutting activities in the first section of this module.

It seems most sensible to do this investigation after students have completed a number of dissection problems, especially if they seem to be a little sloppy in their thinking about the dissections.

Teaching the Investigation

For a timely placement of the Cutting Paradox, use it during or after Investigation 3.2.

The main paradox problem can be done as a short in-class activity, inserted into the work on the module at some opportune moment. Use the activity to motivate discussion of questions like:

- How can we be sure those two pieces come together to create a straight line?
- How can we argue with certainty that we have formed a right angle?
- How can we know that these two edges are the same length so that they fit together without any gaps?

It is in discussion of these issues that students learn much of the geometry in this module.

Continue the class discussion of the paradox problem through the discovery that the "trick" works when the rectangle and square are formed with Fibonacci numbers ("Write and Reflect" Problem 3). Then use Problems 4–9 for homework.

If you have more time and want to explore Fibonacci numbers a little more, pursue the "Perspective on the Fibonacci Sequence" and the "Take It Further" problems.

Teaching Notes

Investigation
3.14

Student Pages 120–123

CUTTING UP SOLIDS

Materials:
- **clay or other modeling material**
- **dental floss or other tools for slicing clay**

Some clear plastic models of the solids are made to allow water inside. As the model is tilted, the water forms a plane surface that shows how a cross section might look.

OVERVIEW

In this investigation, students slice up various solid objects made out of clay, study the resulting cross sections, and then extend their theories about the equidecomposability of plane figures to three-dimensional objects. Can a cube be cut up and rearranged to form another solid? If one solid can be cut and rearranged to form another, what do the two solids have in common? To complete the investigation, a brief history of Hilbert's third problem is presented.

The actual cutting of solids is necessary for most students to visualize cross sections and rearrangements, so preparation for this activity should include testing of the chosen materials to be sure they will actually work for your students. Clay, potatoes, sponges, styrofoam, and gelatin (see "Resources" below for a recipe) have all been used. Having plastic or wooden models of the various solids on hand might also help.

There are no prerequisites for this investigation, but if students have completed Investigation 3.8, they can more fully discuss the equidecomposability of solids.

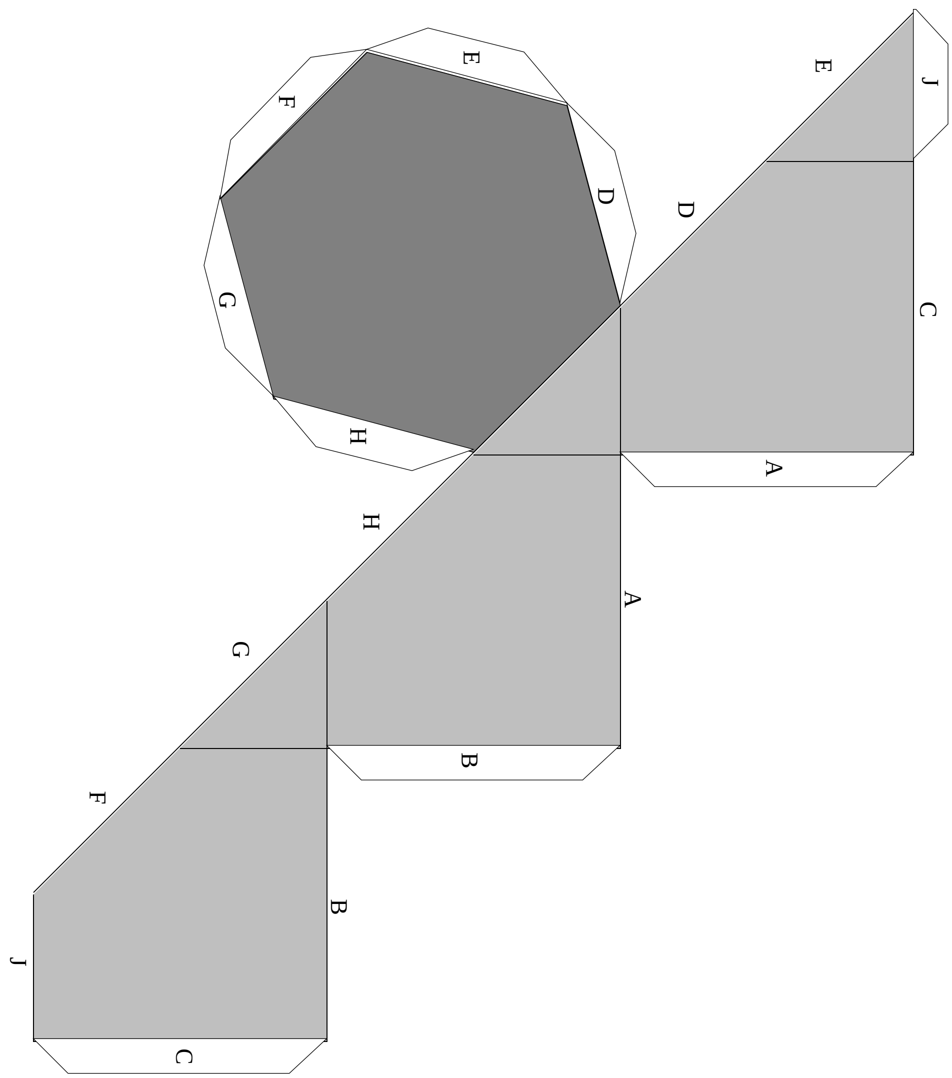
TEACHING THE INVESTIGATION

The cross section problems can be quite time consuming. Visualization is difficult, and even if the objects are actually cut, one must visualize to choose where to make the cuts. Further difficulties are presented when students attempt to draw or describe their results. Consequently, it would be wise to plan some flexibility into your schedule for this investigation.

Problem 3 is difficult; students (and teachers) have a hard time visualizing how to cut a cube to get a pentagonal or hexagonal cross section. The nets below may help.

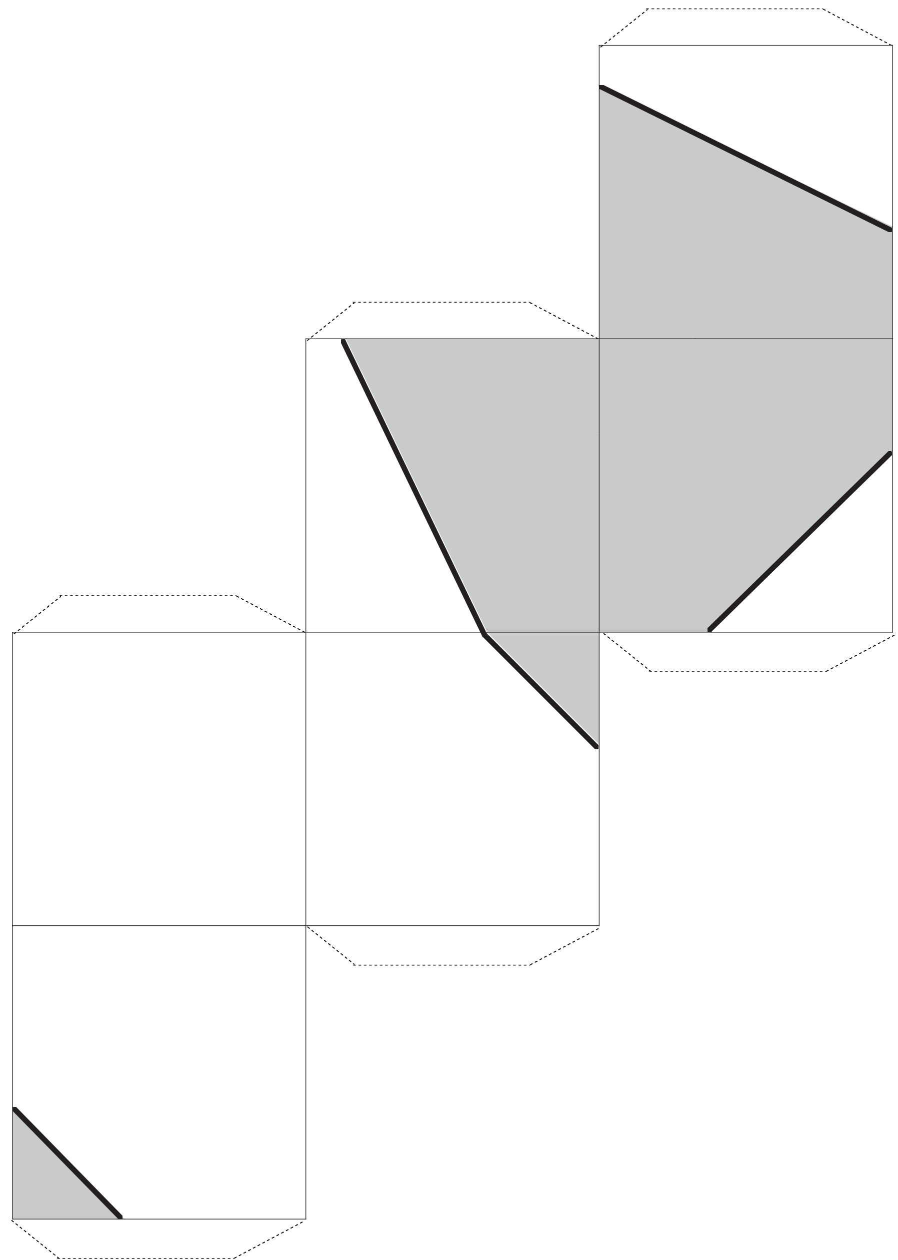
CUTTING UP SOLIDS (continued)

Cut out this picture and fold it up as indicated. Put two of these together to form a cube with a slice through it. The slice forms a hexagonal face.

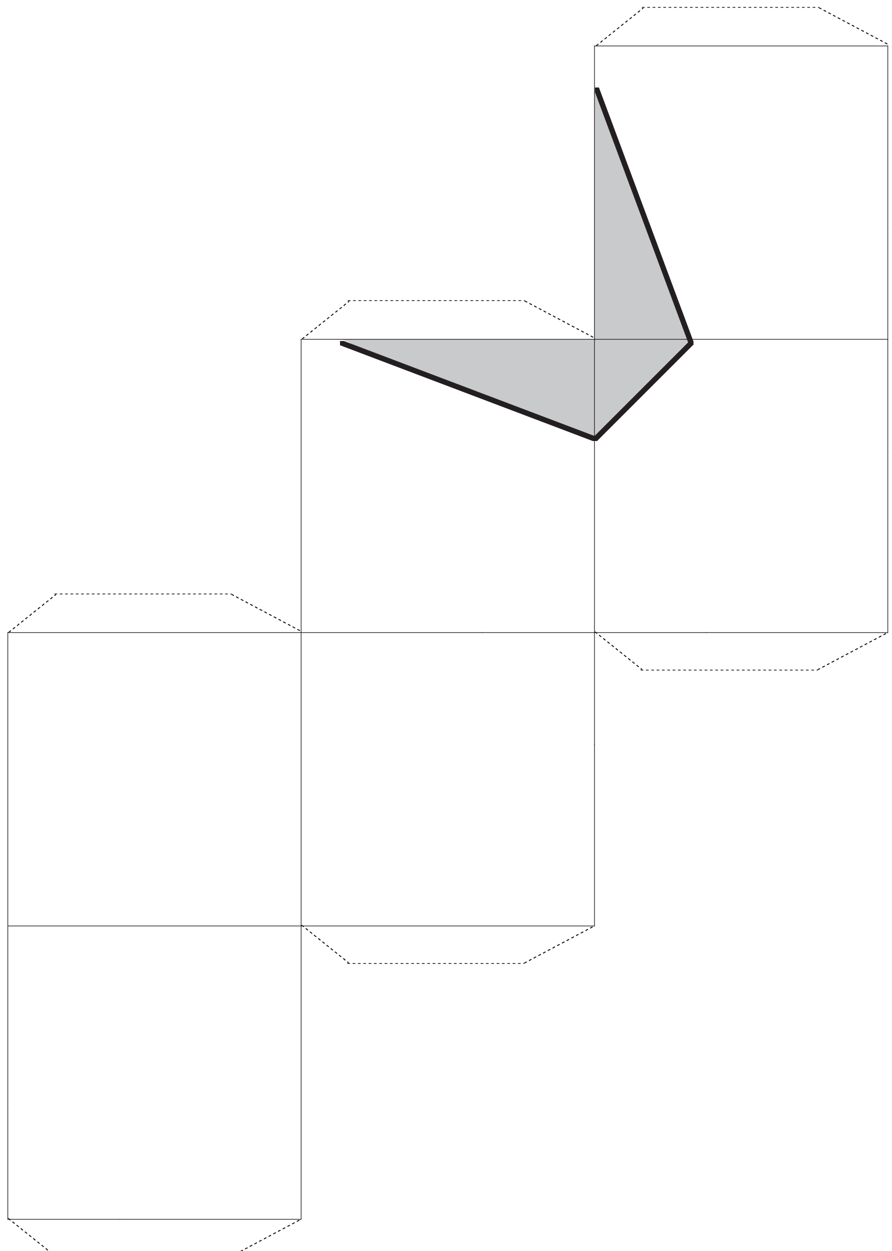


CUTTING UP SOLIDS *(continued)*

This picture folds into a cube with lines marked on it; the lines indicate where to cut to form a pentagonal cross section.



This net folds into a cube with lines indicating how to cut to form a nonequilateral triangle as the cross section.



ADDITIONAL RESOURCES . . . .

Here is a recipe for gelatin from Barney Martinez in Daly City, California. He got it from Bev Bos, a preschool teacher who gives workshops around the country.

1. Mix 32 packets of Knox gelatin with 22 cups of hot water.

2. Use containers of various shapes—cubes, cones, cylinders, or paper cups if that's all you can find—and spray the inside with a vegetable oil spray like Pam.

3. Fill the containers with gelatin, chill, and remove the hardened gelatin from the molds. They will last 2–3 days without refrigeration.

Teaching Notes

Investigation
3.15

Student Pages 124–129

Cut a Rug and Other Dissections

Materials: For the Witch's Hat activity:

- **a generous supply of large, black construction paper**
- **scissors**
- **large compasses**
- **tape**
- **drawing tools**
- **tape measures (to measure head circumference)**

For the Birthday Cake and Checkerboard problems, prepare copies of the objects to be divided so students can plan and carry out the division problems.

Overview

These problems are fun puzzles. The mathematics involved connects well to the content of the module: visualizing a plane surface that will create a solid object and exploring fair division of plane and solid forms.

Teaching the Investigation

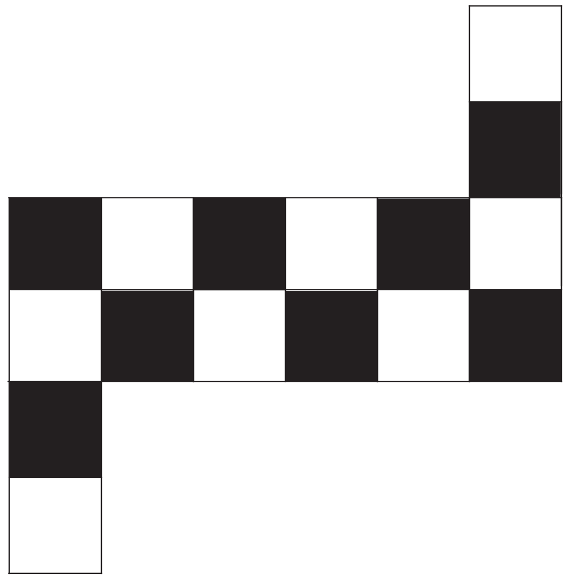
In the witch's hat problem, students will learn what flat shapes can be used to create a cone. The relationship of the dimensions of the hat, the dimensions of the head, and the dimensions of the flat figures can be studied. If there is sufficient time and interest, students could study the cost of producing hats and determine which cutting method uses the least raw material.

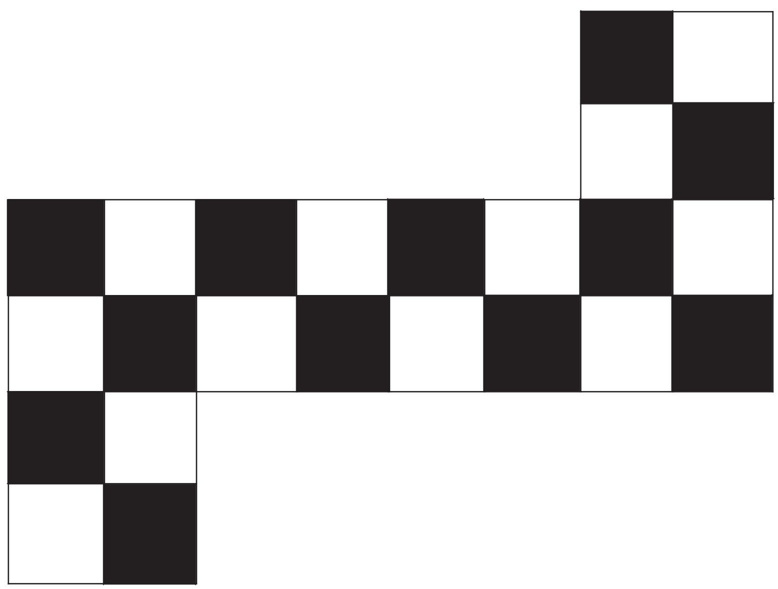
The checkerboard and cake division problems can be used individually as starters, or to fill an entire class on some day when you just want to do something different.

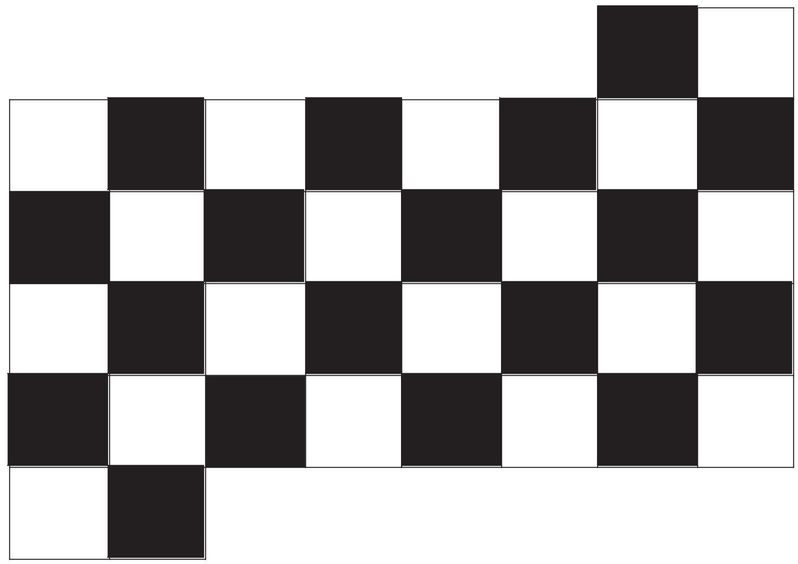
Dividing actual cakes of various shapes can provide refreshments as well as general amusement.

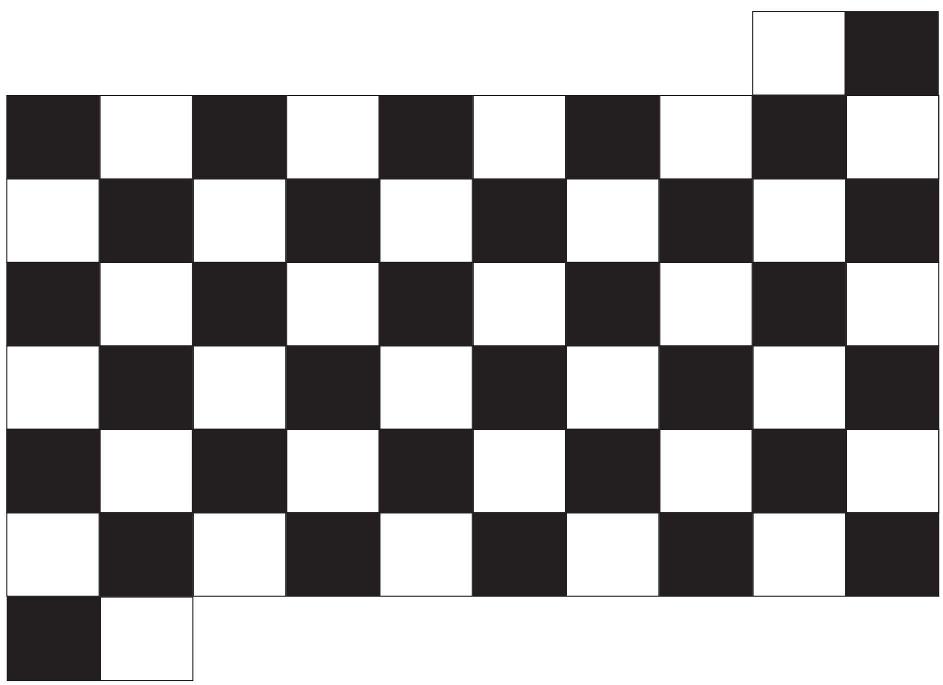
Blackline Masters

Students might find these masters easier to work with, rather than tracing the checkerboards and rugs in the Student Module.

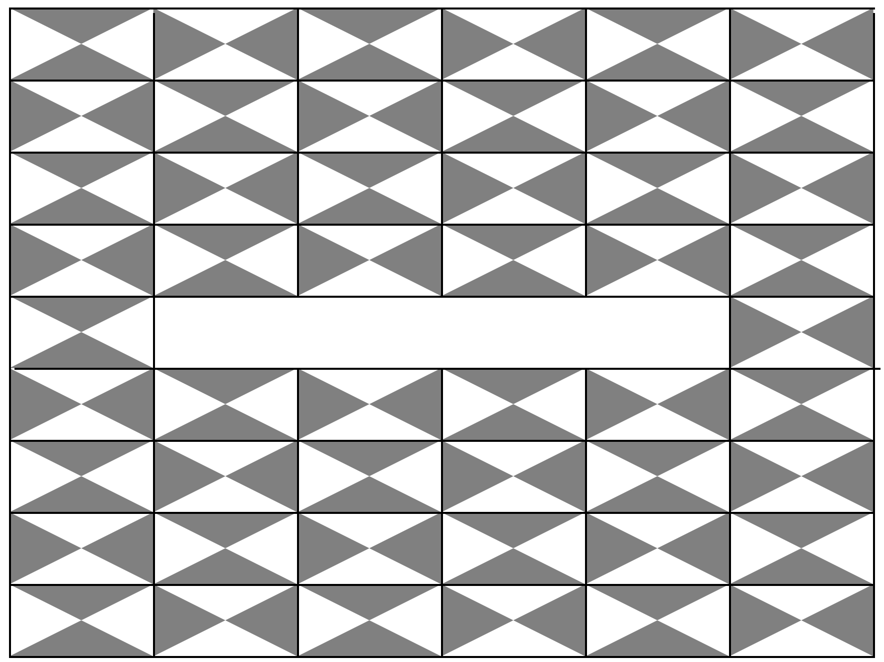


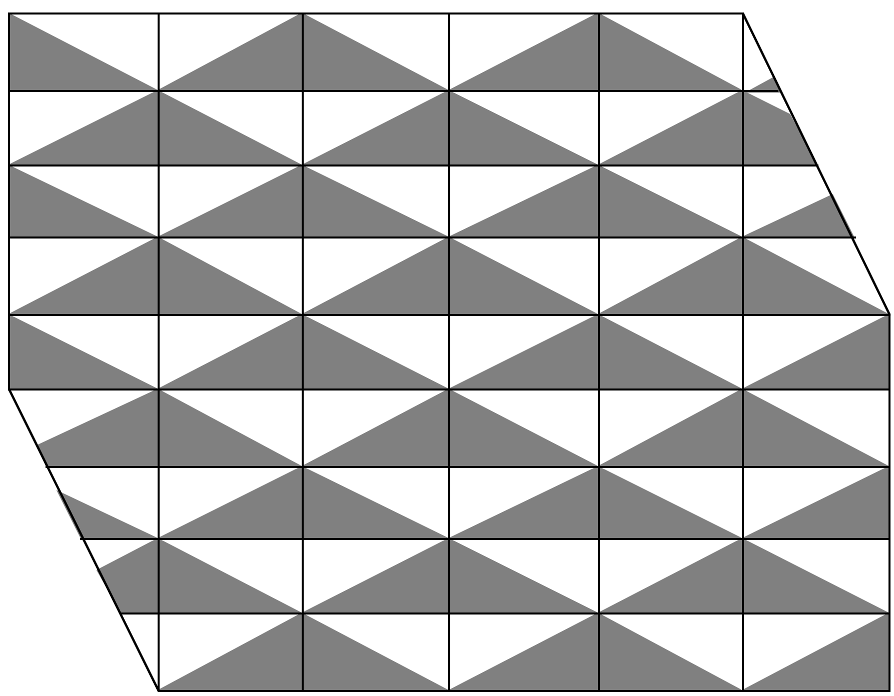






CUT A RUG AND OTHER DISSECTIONS (continued)





Solutions

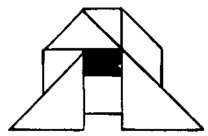
Investigation 3.1

Student Pages 1–9

TANGRAM ACTIVITIES

These pictures are from Ronald C. Read's book *Tangrams: Three Hundred and Thirty Puzzles* (Dover Publications, 1980). Reproduced with permission.

Problem 2 *(Student page 2)* There is more than one way to form each of the letters, but here is one possible solution for each letter A–Z.



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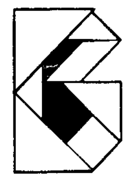

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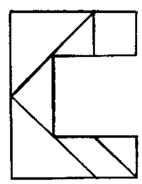

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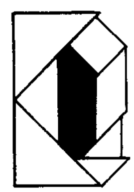

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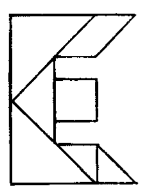

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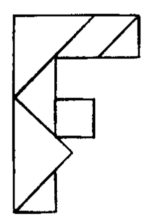

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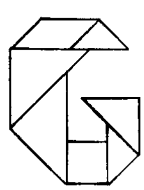

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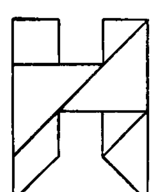

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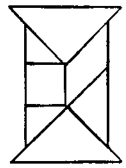
TANGRAM ACTIVITIES *(continued)*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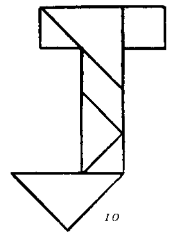
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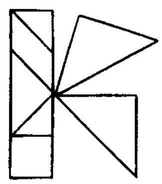
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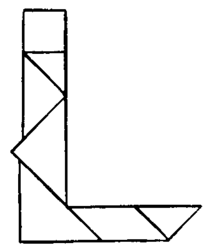
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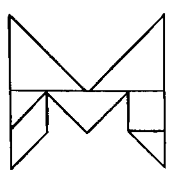
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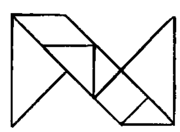
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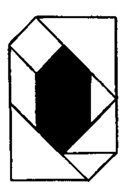
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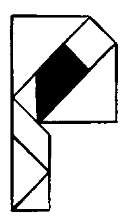
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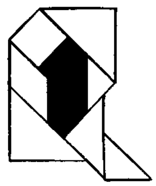
TANGRAM ACTIVITIES *(continued)*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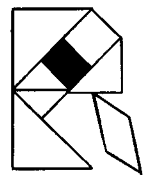
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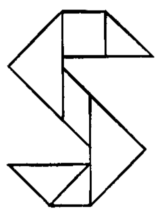
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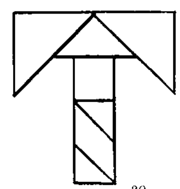
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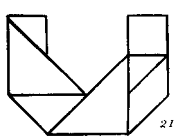
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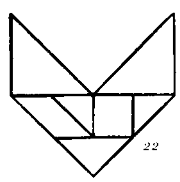
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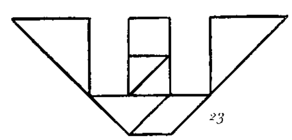
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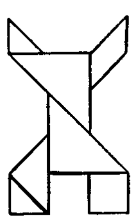
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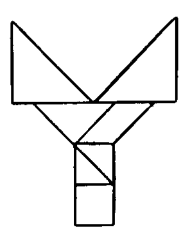
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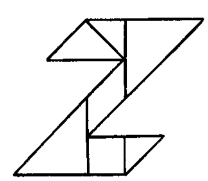
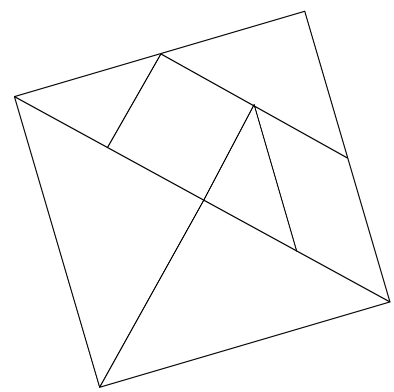
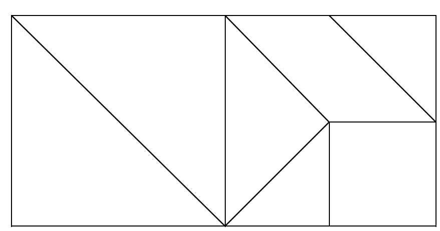

26

Problem 3 *(Student page 2)* All seven tangram pieces fit together to form a square like this:



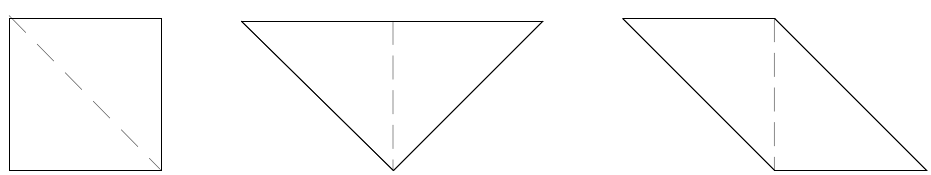
There is at least one other way to do this. Can you find it?

Problem 4 *(Student page 2)* Here's one way to form a nonsquare rectangle, using all seven pieces:



Problem 5 *(Student page 2)* The square and the rectangle above have the same area, as they both consist of the same seven pieces.

Problem 6 *(Student page 2)* The two small triangles can be put together to form the square, middle-size triangle, and parallelogram tangram pieces.



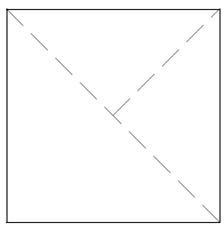
TANGRAM ACTIVITIES *(continued)*

Problem 7 *(Student page 2)*

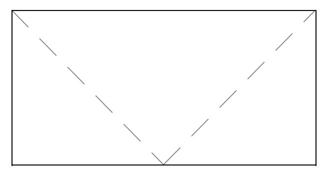
a. Each figure in Problem 6 has twice the area as one of the small triangles. This is because each of these figures consists of two small triangles.

b. Even though the three figures look completely different, by dissecting them into two small triangles you see they have the same area.

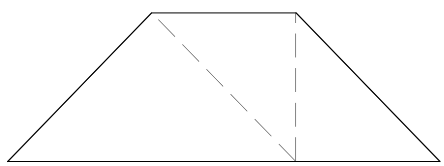
Problems 8–9 *(Student pages 2–3)* Each of the figures below is made up of the two small triangles and the medium-size triangle, so they all have the same area.



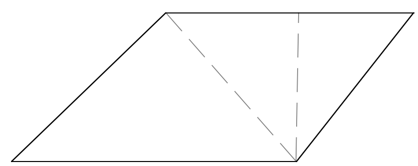
Square



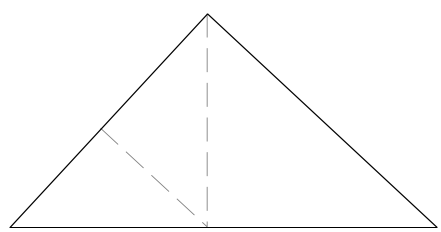
Rectangle



Trapezoid



Parallelogram



Triangle

Problem 10 *(Student page 3)* The square in Problem 8 was made out of the two small triangles and the medium-size triangle. Remember, though, that the medium-size triangle can be dissected into two copies of the small triangle. Thus, the square can also be made out of four small triangles.

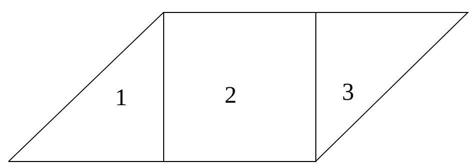
TANGRAM ACTIVITIES *(continued)*

This means that the area of the square is four times the area of one small triangle and twice the area of the medium-size triangle. The square in the tangram set can be made from two of the small triangles, so the square in Problem 8 has twice the area of the tangram square.

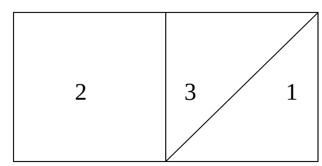
Problem 11 *(Student page 3)* A large tangram triangle can be dissected into four copies of a small triangle, while the medium-size triangle can be dissected into two copies of a small one. Thus, a large triangle has area twice that of the medium-size triangle, and four times that of a small triangle.

For Discussion *(Student page 3)* The goal here is to realize that if two figures can be dissected into congruent pieces, then they have the same area.

Problem 12 *(Student page 5)* For the first pair of pictures, label the pieces of the first figure as 1, 2, 3, as shown below. To create the second figure, translate piece 1 to the right, parallel to the base of square 2.

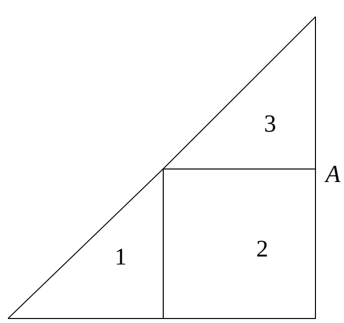


$\mathbf{a_1}$

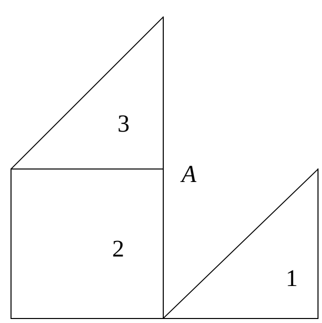


$\mathbf{a_2}$

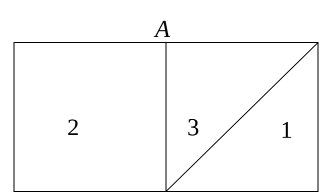
For the second pair of pictures, label the pieces of the first figure, and label point *A* as shown. To form the second figure, translate piece 1 to the right, parallel to the base of piece 2. Then rotate piece 3 about *A* by 180°.



$\mathbf{b_1}$



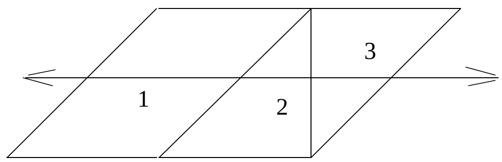
$\mathbf{b_2}$



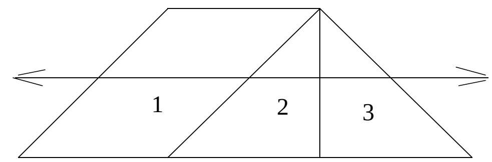
$\mathbf{b_3}$

TANGRAM ACTIVITIES *(continued)*

For the third pair of pictures, once again label the first figure as below. Reflect piece 3 about the line through the midpoints of the sides of the large parallelogram.



c_1



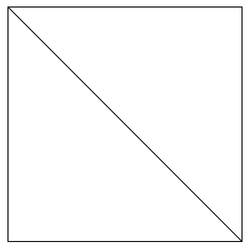
c_2

(For this problem, other solutions are possible.)

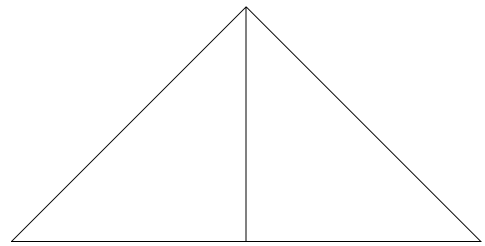
Problem 13 *(Student page 6)* Figures a, b, d, e, and g all have the same area, as they all can be dissected into four copies of the little triangle: Shapes a and d each consist of the parallelogram and the two small triangles. Shapes b, e, and g are made of the square and two small triangles. Notice, however, that both the square and the parallelogram can themselves be dissected into two small triangles.

Shapes c and f are both made from the square, the parallelogram, and two small triangles. Therefore, these two shapes have the same area, which is larger than the area of the other five shapes.

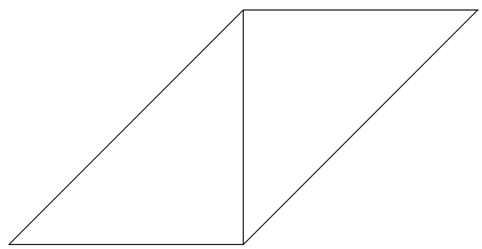
Problem 14 *(Student page 7)* Shapes a, b, and c seem to have the same area. Each can be dissected into two congruent right triangles, and all six triangles shown below are congruent.



a



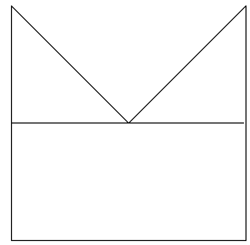
b



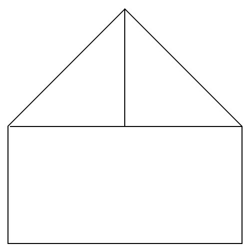
c

Shapes d, e, and f also seem to have the same area as as each other, as they can each be dissected into a rectangle and two congruent smaller triangles where the three

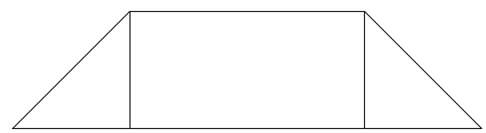
rectangles shown below are all congruent and the six right triangles are all congruent.



d



e



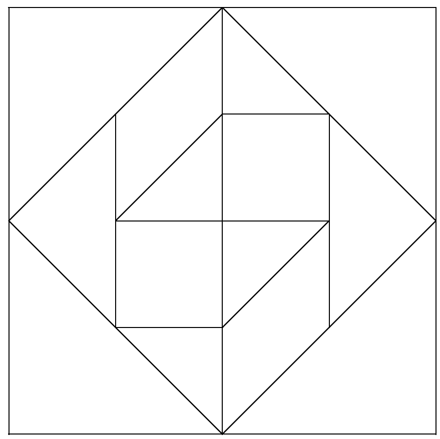
f

Problem 15 *(Student page 7)* If two figures are congruent, they have the same area. Remember that one of our early definitions of congruence was that two figures are congruent if and only if one can fit exactly on top of the other; this means they have the same shape and the same area.

Just because two figures have the same area, however, does not imply that they are congruent. For example a rectangle with dimensions 2″ × 8″ and a square with dimensions 4″ × 4″ both have an area of 16 square inches, but they are not congruent. In the tangrams, the square and the parallelogram have the same area (as they are both made of two small triangles), but they are not congruent.

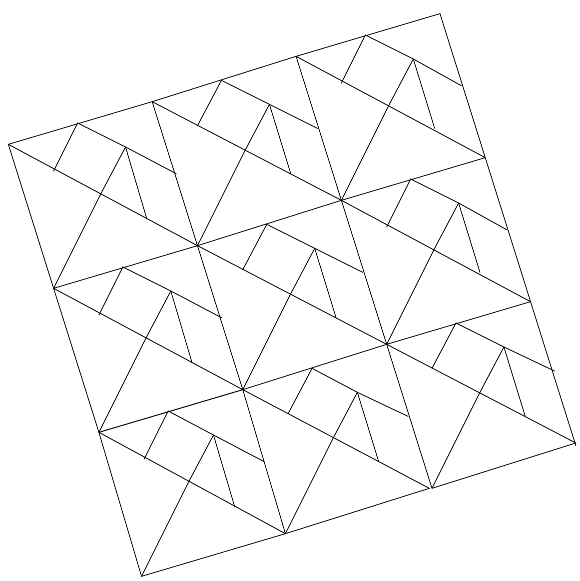
Problem 16 *(Student page 8)*

a. The picture below shows a square consisting of all 14 pieces from two tangram sets. Each set is made into a rectangle whose dimensions are in a 1:2 ratio. Attaching the long side of two such rectangles forms a square.



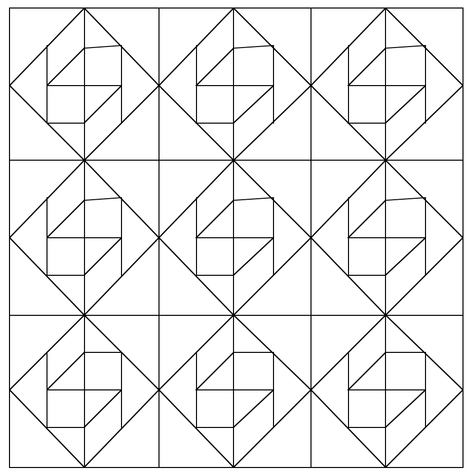
b. If n is a perfect square, or if n is twice a perfect square, then you can make a square using all the pieces from n tangram sets.

If n is a perfect square, take n copies of the square that you made from the pieces of one tangram set, and arrange them in $\sqrt{n}$ rows and $\sqrt{n}$ columns. For example, suppose $n = 9$. Arrange 9 copies of the square as shown below:



TANGRAM ACTIVITIES (continued)

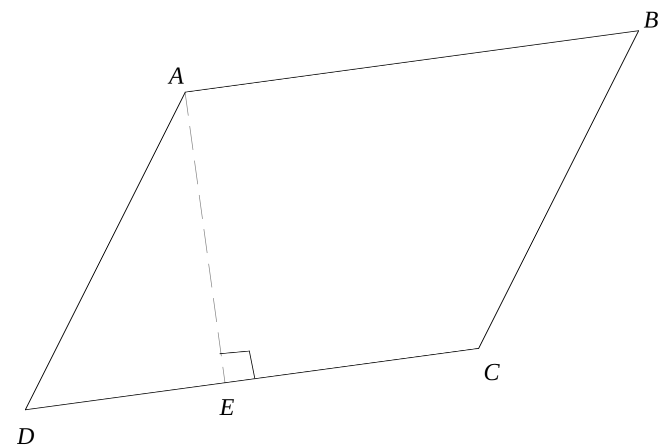
Now suppose n is twice a perfect square; write $n = 2m^2$. Take m^2 copies of the square which you made from two complete tangram sets, and arrange them in m rows and m columns. This will form a square consisting of m^2 smaller squares, but with pieces from $2m^2$ tangram sets. The picture below shows the case where $n = 18$, so $m^2 = 9$.



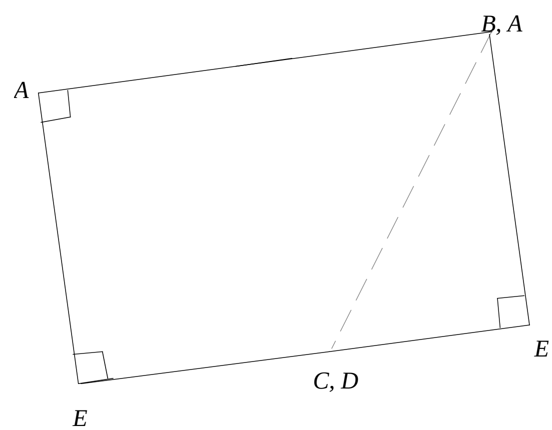
CUT AND REARRANGE

Problem 1 *(Student page 10)* Note that there are many ways to do each of the problems in this investigation. Your methods might differ from those shown here.

Start with parallelogram *ABCD*. Fold over one corner, creating $\overline{AE}$ perpendicular to $\overline{DC}$.



Translate $\triangle ADE$ to the right, along $\overline{EC}$; stop when you can line up segments $\overline{AD}$ and $\overline{BC}$. The resulting figure will be a rectangle.



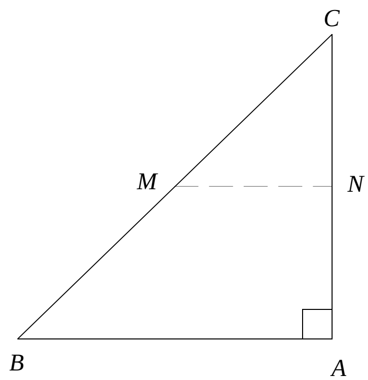
It is worth noting that the perpendicular doesn't have to be drawn from the vertex *A*; it can be drawn from any point on $\overline{AB}$ as long as the perpendicular from that point intersects $\overline{CD}$.

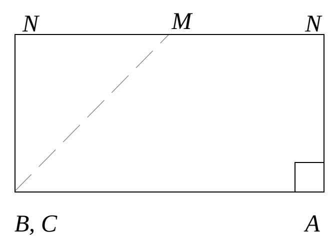
CUT AND REARRANGE *(continued)*

Problem 2 *(Student page 11)* Label the vertices of the triangle A, B, and C, with the right angle at A. Let M and N be the midpoints of $\overline{BC}$ and $\overline{CA}$, respectively. (You can find the midpoints by folding, if you like.)

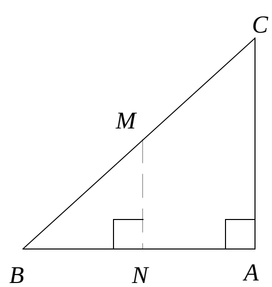
Can you see how you're using the Midline Theorem here?

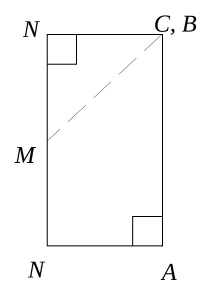
Cut along $\overline{MN}$, and rotate $\triangle MCN$ about M, lining up $\overline{MC}$ with $\overline{MB}$. This will form a rectangle.



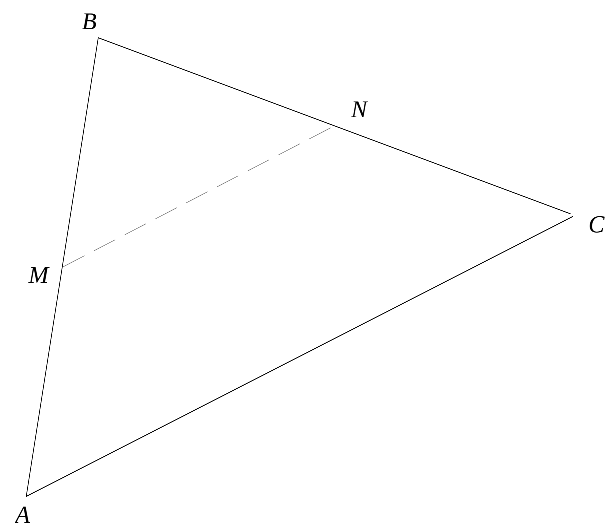


Here's another set of pictures, showing a different orientation of the figures :

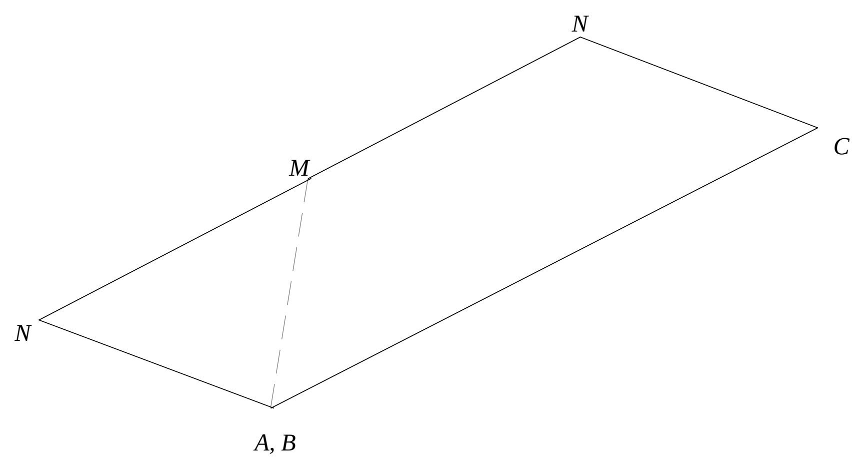




Problem 3 *(Student page 12)* Start with $\triangle ABC$, and let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$.

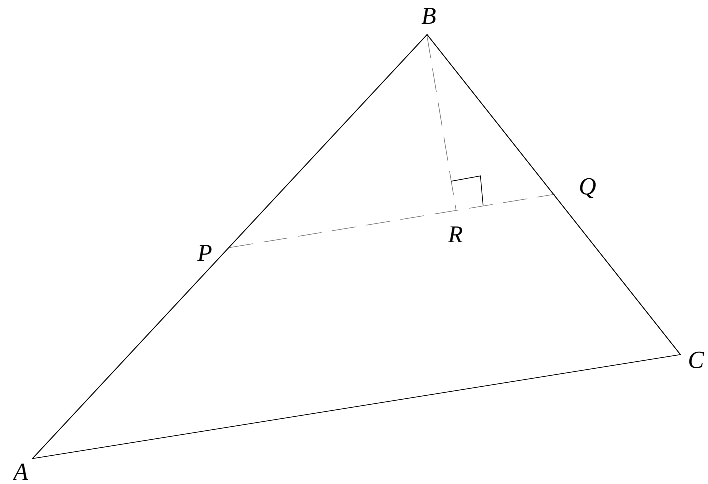


Cut along $\overline{MN}$, and rotate $\triangle BMN$ 180° about M, aligning $\overline{MB}$ with $\overline{MA}$, forming a parallelogram.



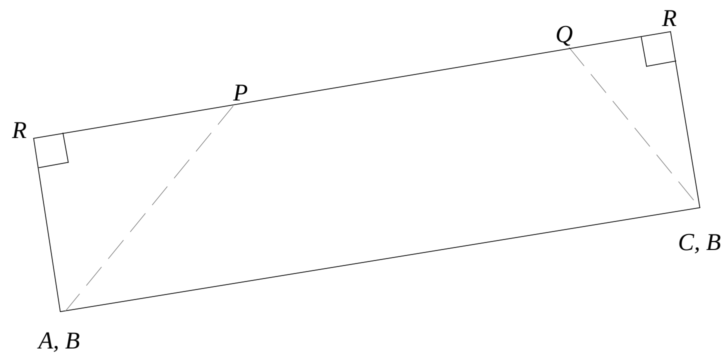
CUT AND REARRANGE *(continued)*

Problem 4 *(Student page 13)* Label the vertices of the triangle A, B, and C, and label the midpoints of $\overline{AB}$ and $\overline{BC}$ as P and Q, respectively. Draw an altitude of $\triangle BQP$ from B, intersecting $\overline{PQ}$ at R.



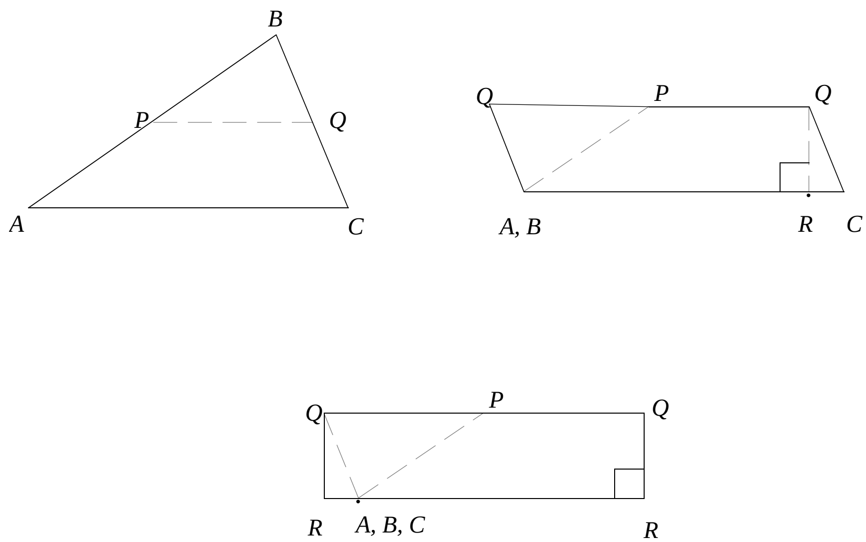
Why is is important that *P* and *Q* are midpoints?

Now, cut along $\overline{PQ}$ and $\overline{BR}$; essentially you want to cut out triangles $\triangle PBR$ and $\triangle QBR$. Rotate $\triangle PBR$ counterclockwise about P, aligning $\overline{PB}$ with $\overline{PA}$, and then rotate $\triangle QBR$ clockwise about Q, aligning $\overline{QB}$ with $\overline{QC}$. This will form a rectangle.

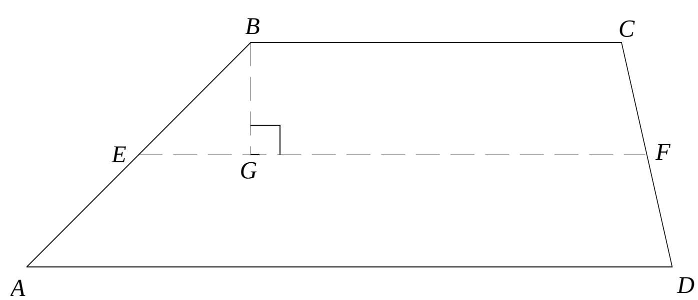


CUT AND REARRANGE (continued)

Another method here is to first transform the triangle into a parallelogram, and then cut and rearrange the parallelogram to form a rectangle; this takes advantage of methods you already know.

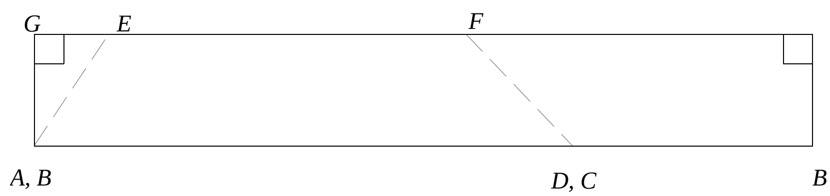


Problem 5 *(Student page 14)* The same method you used for a triangle works here. Begin with trapezoid $ABCD$, and let E and F be the midpoints of $\overline{AB}$ and $\overline{CD}$ (the nonparallel sides), respectively. Draw segment $\overline{EF}$, and draw a segment from B which intersects $\overline{EF}$ perpendicularly at G.

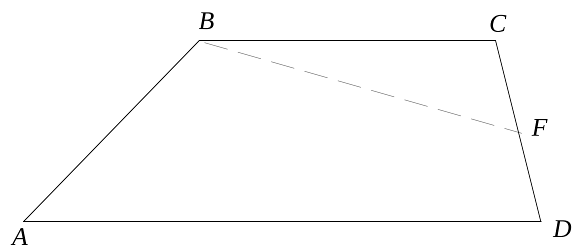


CUT AND REARRANGE *(continued)*

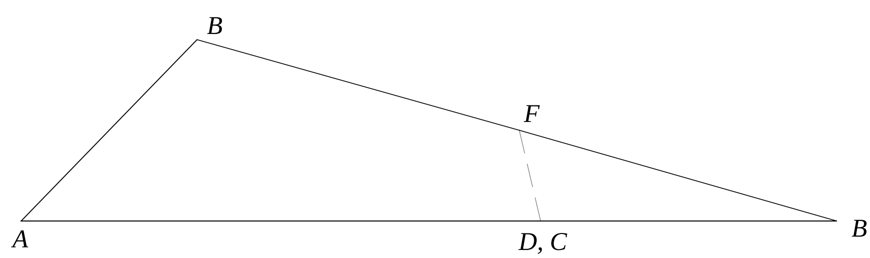
Cut out $\triangle BEG$ and rotate it counterclockwise about E until $\overline{AE}$ and $\overline{BE}$ line up. Then cut out trapezoid *BCFG*, and rotate it clockwise about F until $\overline{FC}$ and $\overline{DC}$ meet. This will draw a rectangle.



Problem 6 *(Student page 14)* This time you want to transform trapezoid *ABCD* into a triangle. Start by letting F be the midpoint of $\overline{CD}$ (one of the nonparallel sides), and then form segment $\overline{BF}$.



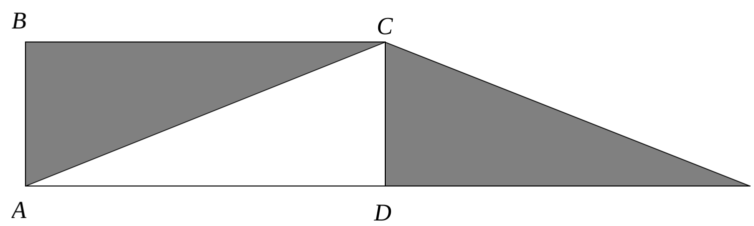
Cut along $\overline{BF}$, rotating $\triangle BCF$ 180° clockwise, lining up $\overline{CF}$ with $\overline{DF}$, and forming a triangle.



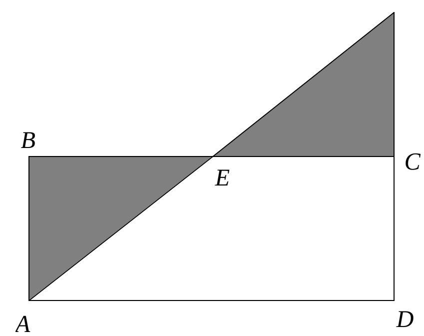
Cut and Rearrange (continued)

Problem 7 *(Student page 14)* In each of the following cases, start with rectangle *ABCD*:

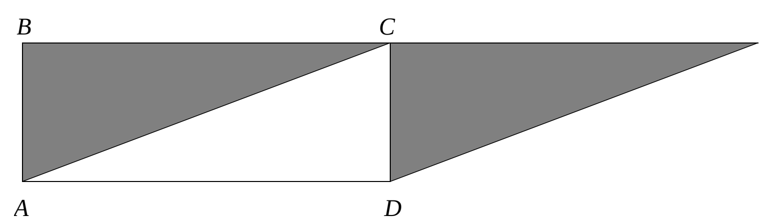
a. To form an isosceles triangle, cut along diagonal $\overline{AC}$, flip $\triangle ABC$ over $\overline{AD}$, and then translate $\triangle ABC$ so that $\overline{AB}$ matches $\overline{CD}$.



b. To form a right triangle, first let E be the midpoint of $\overline{BC}$. Cut along $\overline{AE}$, and rotate $\triangle ABE$ 180° clockwise about E, aligning $\overline{BE}$ with $\overline{EC}$. This will produce a right triangle.



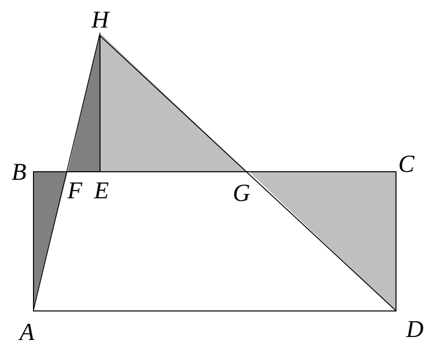
c. To produce a nonrectangular parallelogram, cut along diagonal $\overline{AC}$. Translate $\triangle ABC$ to the right along $\overline{AD}$ until sides $\overline{BA}$ and $\overline{CD}$ match up, producing a parallelogram.



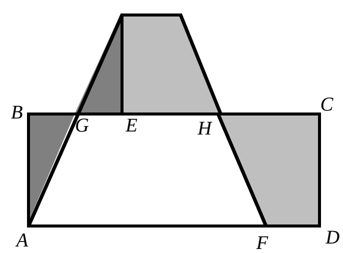
CUT AND REARRANGE *(continued)*

Not every location for *E* will work. If *E* is the midpoint of *BC*, then $AH = DH$. Depending on the dimensions of the rectangle, other locations may give $AH = AD$ or $DH = AD$. But with only three "problem points" along the whole segment, you will probably end up with a scalene triangle.

d. To make a scalene triangle, pick any point E on $\overline{BC}$. Let F and G be the midpoints of $\overline{BE}$ and $\overline{EC}$, respectively. Cut along $\overline{AF}$ and $\overline{DG}$. Rotate $\triangle ABF$ 180° clockwise about F, aligning $\overline{BF}$ with $\overline{EF}$, and rotate $\triangle DCG$ 180° counterclockwise about G, aligning $\overline{GC}$ with $\overline{GE}$. Then $\triangle AHD$ is a scalene triangle made up of the pieces of rectangle $ABCD$.



e. You can make a trapezoid from rectangle $ABCD$ in much the same way you made the scalene triangle. Instead of working from both vertices A and D, pick a point E on $\overline{BC}$ and a point F on $\overline{AD}$. Make G and H the midpoints of $\overline{BE}$ and $\overline{EC}$, respectively.



Draw $\overline{AG}$ and $\overline{FH}$. Now cut out $\triangle ABG$ and rotate it 180° about G, matching $\overline{BG}$ with $\overline{GE}$. Then cut out trapezoid $CDFH$ and rotate it 180° about H, aligning $\overline{CH}$ with $\overline{HE}$. The resulting figure is a trapezoid.

Problem 8 *(Student page 15)* Here are several properties of parallelograms:

- Parallelograms have exactly four sides.
- Opposite sides are parallel.
- Opposite sides are congruent.

- Opposite angles are congruent.
- Consecutive angles are supplementary.
- The diagonals bisect each other.

Problem 9 *(Student page 17)*

a. Because opposite sides of a parallelogram are congruent, you know that

$$AD = BC,$$

so the two sides $\overline{AD}$ and $\overline{BC}$ will fit together exactly.

b. To show that the bottom edge of the new figure will be straight, use the fact that consecutive angles of a parallelogram are supplementary. This tells you that

$$m\angle DAB + m\angle ABC = 180^\circ,$$

so these two angles form a straight line; thus the bottom edge is straight, and the new figure is, indeed, a rectangle.

Problem 10 *(Student page 18)* Since M is the midpoint of $\overline{AC}$ in the original triangle, it follows that $MC = MA$, so sides $\overline{MC}$ and $\overline{MA}$ fit together exactly.

Problem 11 *(Student page 18)* In the original $\triangle ABC$, you know that $m\angle CMN + m\angle NMA = 180^\circ$ since they form a straight line. This implies that the top of the new figure is straight, since these are the two angles which join together to form the top edge.

Problem 12 *(Student page 18)* The side labeled $\overline{AY}$ is really segment $\overline{CN}$ from the original triangle, and the side labeled $\overline{BX}$ is actually $\overline{BN}$ from the original triangle. Since N was chosen to be the midpoint of $\overline{BC}$, it follows that these two lengths are equal, so $\overline{AY} \cong \overline{BX}$.

Problem 13 *(Student page 19)* Here are several properties of rectangles:

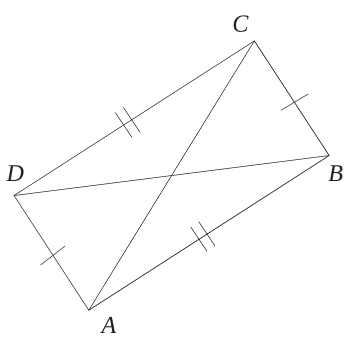
- Rectangles are parallelograms.
- Rectangles have exactly four sides.
- All angles measure 90°.
- Opposite sides are parallel.
- Opposite sides are congruent.
- The diagonals bisect each other.
- The diagonals are congruent.

CUT AND REARRANGE (continued)

Problem 14 *(Student page 20)*

a. You know that $\overline{AD}$ will fit $\overline{BC}$ exactly because they are opposite sides of rectangle *ABCD*, and hence are congruent.

b. Notice that the new top edge is formed by angles $\angle DCB$ and $\angle DAB$; since these are both right angles, their sum is 180°, so they form a straight line. This lets you conclude that the new figure is actually a triangle.



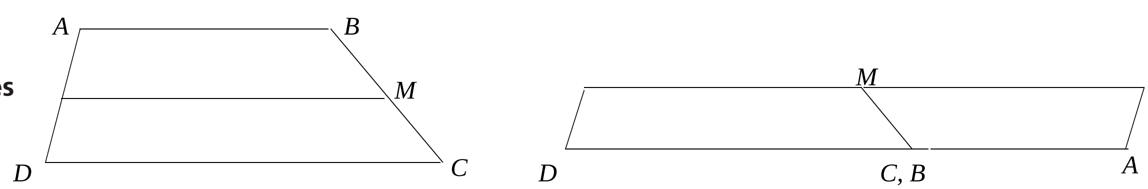
Problem 15 *(Student page 20)* Since the diagonals of a rectangle are congruent, and since opposite sides of a rectangle are congruent, the following is true by the SSS congruence postulate:

$$\triangle BCD \cong \triangle DAB \cong \triangle CBA \cong \triangle ADC.$$

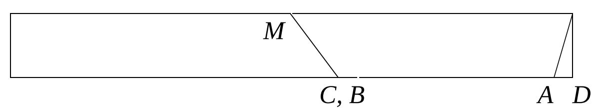
This means that no matter which diagonal you cut along, the two triangles you form will be congruent, so, when joined together, they will form the same isosceles triangle. However, if you slide $\triangle ABD$ up instead of to the right, you will end up with a different isosceles triangle even though it's made of the same two right triangles.

Can you figure out why? How can two congruent right triangles be put together to form two noncongruent isosceles triangles?

Problem 16 *(Student page 20)* Folding $\overline{AB}$ to $\overline{CD}$ created a segment parallel to the two bases through the midpoints of $\overline{AD}$ and $\overline{BC}$. Call the midpoint of $\overline{BC}$ M. After cutting along that segment, the group rotated the top piece about M 180° to form a parallelogram.



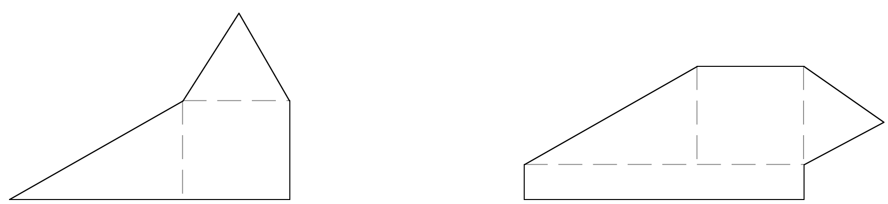
"Folding the left end corner" probably means they folded *D* over to form the small right triangle shown in the picture below. After cutting off that triangle, they translated it along the base $\overline{DC}$, turning the parallelogram into a rectangle by forming adjacent 90° angles (making "a straight end").



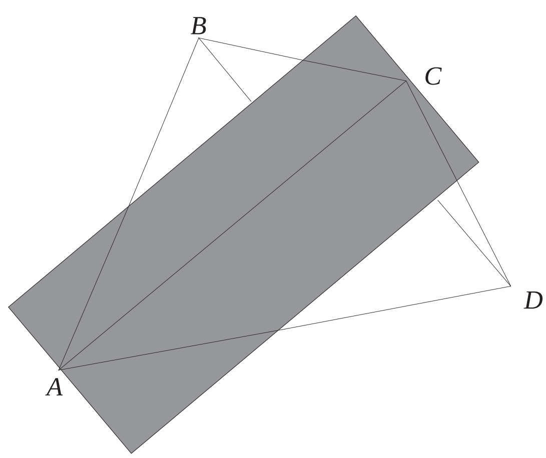
Cut and Rearrange (continued)

Problem 17 *(Student page 21)* Suppose you have two figures, and you want to show that one has a larger area than the other. If you can cut up one figure so that it fits entirely inside of another and there is some of the second figure left over (not covered by pieces of the first), then the second figure must have the larger area.

In the picture below, the first figure can be dissected into three pieces: a square and two triangles. The second figure can be dissected into four pieces; the square and two triangles from the first figure, plus an additional rectangle. This shows that the second figure has the larger area.

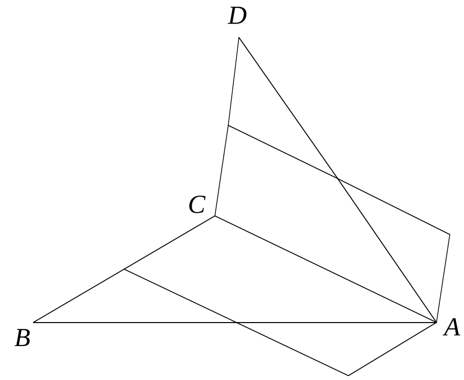


Problem 18 *(Student page 22)* Call the first quadrilateral *ABCD*, and draw diagonal $\overline{AC}$. This forms two triangles, $\triangle ABC$ and $\triangle ADC$, which share base $\overline{AC}$. Dissect each triangle separately into a rectangle with base $\overline{AC}$. The two rectangles will align to form a larger rectangle with the same area as *ABCD*. The picture below shows this process.

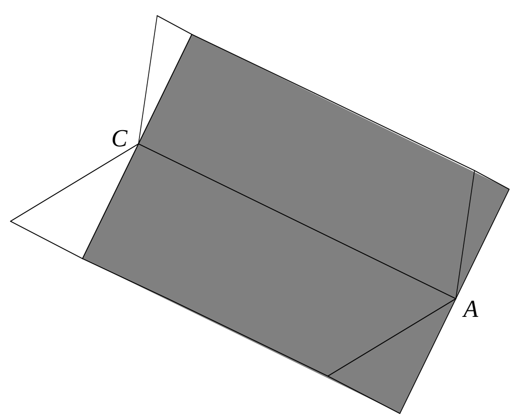


The method used above will not work for the concave quadrilateral. There is only one diagonal you can draw which lies inside the figure, and this diagonal ($\overline{AC}$) creates two obtuse triangles. The method used above to dissect a triangle into a rectangle fails in an obtuse triangle unless the base is the longest side.

For the concave quadrilateral, first draw diagonal $\overline{AC}$ to divide the quadrilateral into two triangles, as before. Next, transform each triangle into a parallelogram by joining the midpoints of the two sides of each triangle that are not the common side $\overline{AC}$.

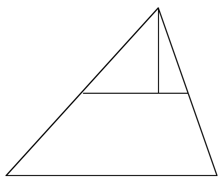


Now transform each parallelogram into a rectangle which has $\overline{AC}$ as a base. These two rectangles together form one larger rectangle which has the same area as *ABCD*.



Problem 19 *(Student page 22)* Because you had to use two different methods in the two previous problems, it is not clear at this point whether you could write down an algorithm to change *any* quadrilateral into a rectangle. You do know, however, that you can divide a quadrilateral into two triangles, and that each triangle can be made into a rectangle. What is not clear is when this can be done so that these two rectangles share the same base, and can hence be joined together to form one larger rectangle.

CUTTING ALGORITHMS



Problem 1 *(Student page 24)* Find the midpoints of two sides of the triangle and connect them. Then draw an altitude from the vertex opposite the base of the triangle to the midline you just drew. This will form two small right triangles contained inside the larger triangle. Each of these triangles has one of the midpoints constructed earlier as a vertex. Rotate each triangle by 180° about this corresponding midpoint. The hypotenuse of each right triangle will line up against the lower half of a side of the original triangle, and you will form a rectangle.

Problem 2 *(Student page 24)* Here are algorithms to accompany the pictures from the solutions for Problems 1, 4 and 5 in Investigation 3.2. We no longer refer to labeled vertices of the polygons because we want someone to be able to perform these manipulations without looking at the pictures.

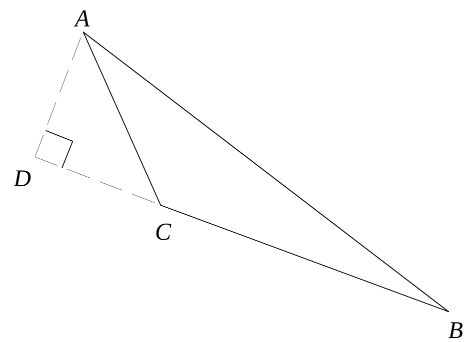
- To dissect a parallelogram into a rectangle: First orient the parallelogram so that you can drop an altitude from one of the top vertices that passes through the opposite side, and then construct such an altitude. This altitude creates a right triangle that has a side of the parallelogram as its hypotenuse. Cut along the altitude; translate this right triangle to the opposite side of the parallelogram, aligning its hypotenuse with the corresponding parallel opposite side.

- To dissect a triangle into a rectangle: If the triangle has an obtuse angle, place the triangle so that the side opposite the obtuse angle is used as the base. (Otherwise, choose any side as your base.) Now the algorithm from the solution for Problem 1 above will work to construct a rectangle.

- To dissect a trapezoid into a rectangle: Orient the trapezoid so that the larger of the two parallel bases is on the bottom. Connect the midpoints of the nonparallel sides. This creates two trapezoids, one on top of the other. Rotate the smaller (top) trapezoid 180° about one of the midpoints you connected. This forms a parallelogram. Now follow your algorithm to change this parallelogram into a rectangle.

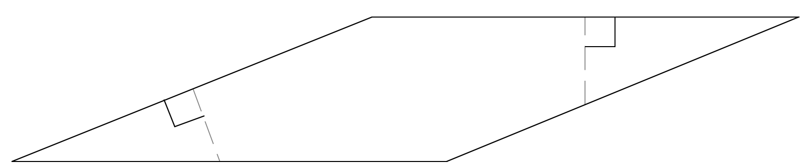
An altitude to the base of an isosceles triangle is also a median. This is why the algorithm works for isosceles triangles.

Problem 6 *(Student page 26)* This method only works for isosceles triangles. There are two main problems with this algorithm. First, it assumes that you can pick any vertex of the triangle, and draw an altitude from it to the opposite side. In many triangles, such as obtuse $\triangle ABC$ below, this is not possible; the line through vertex A perpendicular to the opposite side, $\overline{BC}$, does not intersect the side at all.



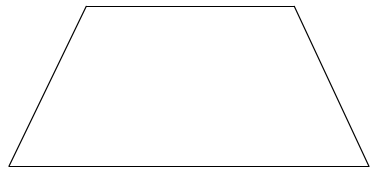
Moreover, it assumes that sides of the triangle are congruent when it matches them up in the third step.

Problem 7 *(Student page 27)* This algorithm only fails in an extreme case, but, nevertheless, it can't be considered an acceptable algorithm unless in works for *all* parallelograms. The problem is that the perpendicular bisector of a side of the parallelogram might not pass through the opposite side. In the picture below, two perpendicular bisectors are shown, each of which actually passes through the *adjacent* side. In fact, you can test the other two sides as well, and see that none of the perpendicular bisectors passes through the opposite side of the parallelogram.

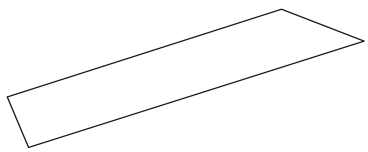


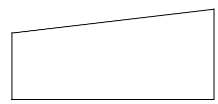
To fix the algorithm, start it this way: "Orient the parallelogram so that some perpendicular drawn from a vertex on the top side will intersect the bottom side. Now cut along that perpendicular, which guarantees right angles." (The rest of the given algorithm will work.)

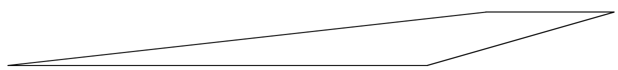
For Discussion *(Student page 29)* By definition, a trapezoid is just a quadrilateral with exactly one pair of parallel sides. Most people imagine a trapezoid looking like this:



The following figures are also all trapezoids, even though they are nonstandard:







Problems 9–10 *(Student page 30)* You need to check at least two properties in order to show a shape is a rectangle. Some properties which suffice to guarantee a figure is a rectangle are the following:

- A quadrilateral with opposite sides congruent and one right angle;
- A quadrilateral with all angles measuring 90°;
  (In fact, three angles are enough. Why?)
- A quadrilateral with opposite sides parallel and one right angle;
- A parallelogram with congruent diagonals.

It may be hard to show that two sides are parallel by cutting and rearranging; lengths and angles are easier to check, especially if you keep track of the cuts you make (such as when you were cutting along an altitude or a midline).

CUTTING ALGORITHMS *(continued)*

Problem 12 *(Student page 31)*

a. An *algorithm* is a predetermined process, or set of steps, designed to perform a task or to solve a problem. The importance of an algorithm is that each time you implement it, it should work the same way.

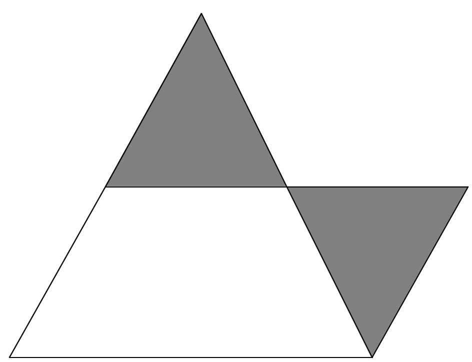
b. A *trapezoid* is a quadrilateral with exactly one pair of sides parallel.

c. A *dissection* is a way to cut up a figure into smaller pieces.

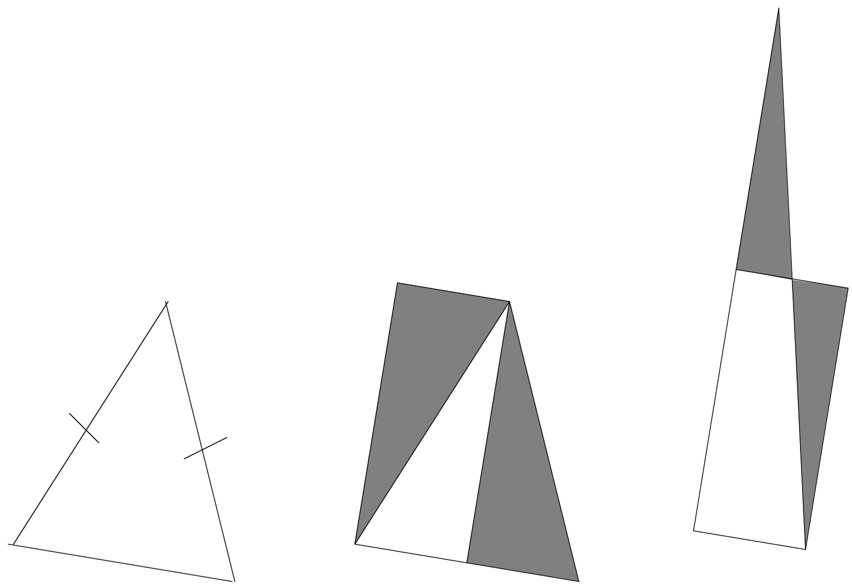
d. A *perpendicular bisector* of a segment is a line that intersects the segment at its midpoint and is perpendicular to the segment.

e. To *debug* means to find and fix any problems. This word is usually used in connection with algorithms and computer programs.

Problem 13 *(Student page 31)* Construct the midpoint of the top side of the parallelogram, and draw a line segment connecting this midpoint to one of the two base vertices of the parallelogram. This will form a triangle and a trapezoid inside the original parallelogram. Rotate the triangle 180° about the midpoint of the top side of the parallelogram. The triangle will now sit on top of the trapezoid, forming a larger triangle.



Problem 14 *(Student page 31)* One way to do this is to turn the triangle into a rectangle and then turn the rectangle into a new triangle:

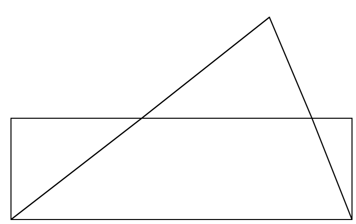


Problem 15 *(Student page 31)* This is simply the "identity transformation"—you don't cut the shape at all.

In the language of functions, the triangle-to-rectangle transformation is not one-to-one.

Problem 16 *(Student page 31)* In general, you cannot recover your original triangle. The triangle-to-rectangle cutting algorithm turns many triangles (all those with the same base and height) into the same rectangle, so there is no way to recover one particular triangle once it has been transformed.

However, if you have a copy of the original triangle to help you, you can reverse the algorithm this way: The triangle and rectangle will both have a side of the same length. Place the triangle on top of the rectangle, aligning these two sides.



This forms two smaller triangles in the corners of the rectangle—the regions that fall inside the rectangle but outside the triangle. Cut out these two triangles, and rotate

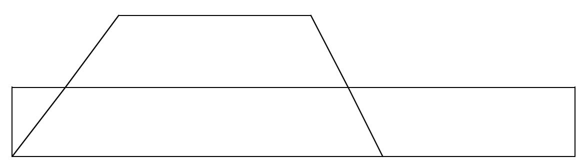
each of them 180° so that they fit together inside the portion of the big triangle which lies outside the rectangle. This is how you dissect the rectangle back into the triangle.

If you don't care about recovering your *particular* triangle, but just want to turn the rectangle into *some* triangle, you can use the algorithm from Investigation 3.2, Problem 7a, b, and d.

In the language of functions, the trapezoid-to-rectangle transformation is not one-to-one.

Problem 17 *(Student page 31)* In general, you cannot recover your original trapezoid. The trapezoid-to-rectangle cutting algorithm turns many trapezoids (all those with the same sum of bases and the same height) into the same rectangle, so there's no way to recover one particular trapezoid once it has been transformed.

However, if you have a copy of the original trapezoid to help you, you can reverse the algorithm by using the same idea as in the solution for Problem 16: Take two copies of the trapezoid. Follow the algorithm to transform one of them into a rectangle. Then, place the other trapezoid inside this rectangle. Look for the pieces of the rectangle which lie outside the trapezoid. These are the pieces which must be cut and rotated to fit inside the trapezoid, giving an algorithm for dissecting a rectangle into a trapezoid.



If you don't care about recovering your *particular* trapezoid, but just want to turn the rectangle into *some* trapezoid, you can use the algorithm from Investigation 3.2, Problem 7e.

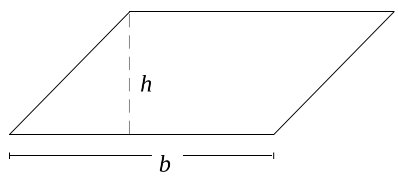
Problem 18 *(Student page 31)* If you have an algorithm for cutting a trapezoid into a rectangle, and another for cutting a rectangle into a parallelogram, you can just combine the two to create an algorithm for cutting a trapezoid into a parallelogram.

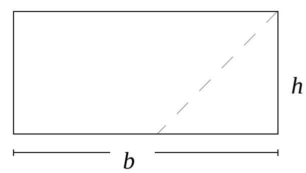
Problem 19 *(Student page 31)* Now suppose you have an algorithm for cutting a trapezoid into a rectangle; one for cutting a triangle into a rectangle, and you want to cut a trapezoid into a triangle. First, undo the triangle-to-rectangle algorithm, as in Problem 16; this creates an algorithm for transforming a rectangle into triangle. By combining this algorithm with the one which cuts a trapezoid into a rectangle, you can go from a trapezoid into a triangle.

Of course, you may be able to come up with a more elegant, direct solution, like the one for Problem 6 of Investigation 3.2, but this shows that if you know two algorithms you can come up with a third that will work.

Area Formulas

Problems 1–3 *(Student pages 32–33)* When you dissect a parallelogram into a rectangle, you cut along an altitude drawn to one side of the parallelogram. The length of this altitude then becomes the height of the rectangle you form. Moreover, the side of the parallelogram to which this altitude is drawn has the same length as the base of the rectangle you form. This means that if a parallelogram has an altitude with length h to a side of length b, then the rectangle obtained by cutting along this altitude will have dimensions $b \times h$.

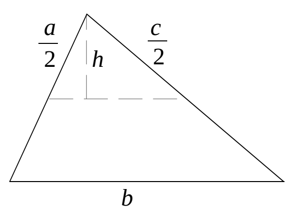


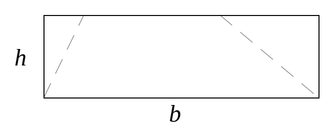


We will assume the area formula for a rectangle and use this to derive all other area formulas.

The parallelogram and rectangle have the same area. You know that the area of the rectangle is given by bh ("base times height" or "length times width"), so the parallelogram must also have area bh. In other words, the area of a parallelogram is given by "base times height," where the height measures the distance from the base to the side parallel to it.

Problems 4–6 *(Student pages 33–34)* Remember that to dissect a triangle to a rectangle you first connect the midpoints of two of the sides, and then draw an altitude to this midline. (Remember that an obtuse triangle must be positioned so that the longest side is the base.) Let's perform the algorithm, starting with a triangle that has sides of length a, b, and c, and let h denote the length of the altitude to the midline.





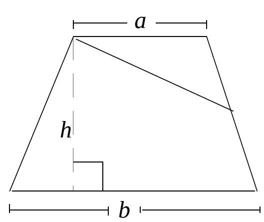
It appears that the dimensions of the resulting rectangle are $b \times h$, where b is the length of the base of the triangle (the side that was not cut).

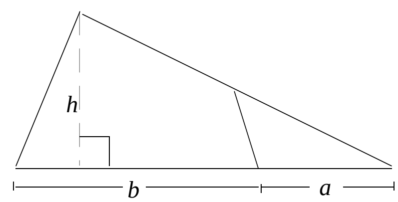
AREA FORMULAS (continued)

The fact that the midline cuts the height in half will be proved later.

Notice that the height from the vertex to the midline is exactly half the length of the altitude of the triangle from that vertex. So if H is the length of the altitude of the triangle to the side having length c, then $h = \frac{1}{2}H$. The resulting rectangle has an area equal to bh, so it follows that the area of the triangle is the same. In other words, the area of the triangle can be expressed as $\frac{1}{2}bH$, or "one half base times height."

Problems 7–8 *(Student pages 35–36)* Here's a way to find the area of a trapezoid by dissecting it into a triangle. Let a and b be the lengths of the two parallel sides of the trapezoid, and let h be the distance between these two sides (the height of the trapezoid).





The resulting triangle has a base of length $a + b$ and a height of length h. Since the trapezoid and the triangle have the same area, and you know how to find the area of a triangle, you can conclude that the area of the trapezoid is given by $\frac{1}{2}h(a + b)$.

For Discussion *(Student page 36)* It is not possible to dissect a circle into a rectangle in a finite number of cuts. This is basically because a circle is not made up of line segments. We will see below that you can dissect the circle into figures which are *almost* rectangles, but they will always have some edges which consist of arcs, not line segments.

Problem 9 *(Student page 38)* The main idea is that each shape looks a little bit more like a rectangle than the one before it. At each step, the resulting shape will always have two scalloped sides and two straight sides, but, as you continue, the scalloped sides will become straighter and straighter, and the straight sides will come close to being perpendicular to the scalloped sides.

You could say that the *limiting shape* of this infinite process is a rectangle.

Even though you cannot actually continue this process indefinitely, you can *imagine* it continuing endlessly, with the shapes becoming more and more like rectangles. Remember that each shape has the same area as the circle, so if the shapes approach a figure whose area you know, you may be able to use this to compute the area of the circle.

AREA FORMULAS (continued)

Problem 10 *(Student page 38)*

a. Each of the two scalloped sides of *Shape 3* is made up of 8 curved pieces (each curved piece being an arc of the circle). The 16 small arcs together form the entire circle. Thus, each scalloped side has a length of $\frac{1}{2}C$, where C is the circumference of the original circle.

b. Each straight side in *Shape 3* comes from a radius of the circle, so each straight side has length r, where r is the radius of the original circle.

Problem 11 *(Student page 38)* Each shape will consist of two scalloped sides and two straight sides. The straight sides will always have length r, and the scalloped sides will always have length $\frac{1}{2}C$.

The number of wedges will always increase; in fact, the nth shape will be made up of 2^{n+1} wedges. Furthermore, the size of each wedge will always decrease; the nth wedge will be equal to $\frac{1}{2^{n+1}}$ of the entire circle. Each figure in the series has the same area—the area of the original circle.

Problem 12 *(Student page 39)*

a. The sides appear parallel because of the opposite angles. The wedges are all congruent, so the opposite angles of the figure are congruent. This makes it look like a parallelogram. As the sides become less scalloped, it looks even more like one.

b. A sequence is just an infinitely long list of objects, so all these shapes form a sequence. Since the shapes start to look more and more like rectangles the farther out in the list you go, you say the sequence "approaches" a rectangle. You never actually get a rectangle, however, because the process never stops.

As the process continues, each wedge of the circle gets narrower and narrower. Eventually, the arc of the wedge starts to look like a straight line; the arc becomes so short that you cannot see its curved shape. Also, the wedge will get so narrow that the sides will look perpendicular to the arc. This is why it begins to look as if there are right angles at each "corner" of the shape, as in a rectangle.

Problem 13 *(Student page 39)* Since each shape has a straight side of length r, and a scalloped side of length $\frac{1}{2}C$, these would be the dimensions of the rectangle. Thus, the "almost" rectangle has an area of $\frac{1}{2}Cr$.

Problem 14 *(Student page 39)* Since π is defined by the formula $C = 2\pi r$, it follows that the "almost rectangle" has an area of $\frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2$. Therefore, you can conclude that the area of a circle with radius r is πr^2.

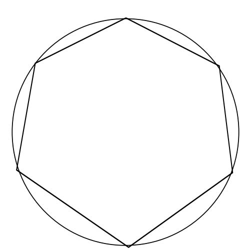
For Discussion *(Student page 40)* The lengths of the sides of the various shapes are precisely $\frac{1}{2}C$ and r. Moreover, the number π is defined precisely; it is the ratio of C to r. The place where the above argument is weak is where we make the jump from the scalloped shapes to the rectangle. This step requires a bit of justification.

The problem doesn't have to end here. You can come back to this later to try to improve your argument, perhaps after learning more mathematics.

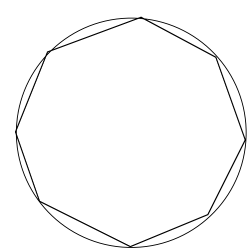
The strength of the argument is that it is convincing, and it provides you with some insight. Even if you're not completely convinced of its validity, you at least have some knowledge of where the area formula for a circle comes from.

Problem 15 *(Student page 40)* One way to measure circumference is to take a piece of string and wrap it around the circle. You can then straighten the string and measure it with a ruler.

Problem 16 *(Student page 40)* You can also approximate the circumference of the circle with straight line segments. The line segments can then be measured with a ruler. To get a better approximation of the circumference, use a larger number of smaller segments.



good approximation



better approximation

These solutions only work if each shape is what it "looks like," for example, if shape 1 is a rectangle. Otherwise, you have no way to judge the relative areas.

Problem 17 *(Student page 40)*

a.–b. Shapes 2, 3, 6, and 7 all have the same area. Shapes 1, 4, and 5 also have the same area, and their area is twice the area of the other four figures.

c. Shapes 1 and 4 have the same height and base measurements. Thus, since the area of both a parallelogram and a rectangle is given by base times height, the two figures have the same area.

d. Shapes 1 and 3 also have the same length base and height. But, since the

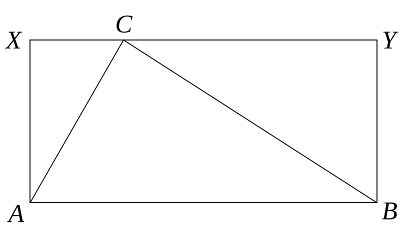
area of a triangle is one half the base times the height, it follows that shape 3 has one half the area of shape 1.

e. Shapes 2 and 3 also have the same base and height measurements (notice that the altitude of shape 2 falls outside of the triangle). Since they are both triangles, the two shapes have the same area.

f. Shapes 2 and 5 have the same height. Moreover, the length of the base of shape 2 is the average of the lengths of the two parallel sides of shape 5. Since the area of a trapezoid is the average of the bases times the height, and the area of a triangle is one half base times height, the area of shape 2 is one half the area of shape 5.

g. Shapes 6 and 7 have the same length bases, although the height of shape 6 is twice the height of shape 7. This implies that they have the same area, as shape 6 is a triangle and shape 7 is a rectangle.

h. Shapes 2 and 7 have the same length base, while the height of shape 2 is twice the height of shape 7. Since the shape with the bigger height is a triangle, it follows that they have the same area.



Problem 18 *(Student page 42)* The sum of the areas of the two smaller triangles is equal to the area of the larger triangle. Since $ABYX$ is a rectangle, $AB = XY$ and the heights of all three triangles are the same, namely the length AX.

$$\text{Area}(\triangle AXC) = \frac{1}{2} XC \cdot h$$
$$\text{Area}(\triangle BYC) = \frac{1}{2} YC \cdot h$$

Adding, we see that the combined area is $\frac{1}{2}(XC + YC) \cdot h = \frac{1}{2} AB \cdot h$, which is exactly the area of triangle $\triangle ABC$.

Problem 19 *(Student page 42)*

a. The height from A to $\overline{BC}$ is 4 cm.

b. The area of the triangle is 6 cm^2.

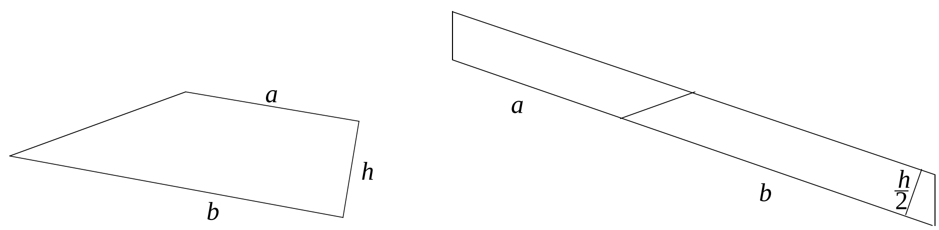
Problem 20 *(Student page 43)* The area of $\triangle ABC$ can be calculated in two different ways. If you think of $\overline{AC}$ as the base, then the height is h, so the area is $\frac{1}{2} h \cdot AC$.

If you think of $\overline{BC}$ as the base, then $\overline{AB}$ is the corresponding altitude, since $\angle B$ is a right angle. So the area of $\triangle ABC$ can also be expressed as $\frac{1}{2} AB \cdot BC$.

AREA FORMULAS (continued)

Since the area of a triangle is independent of how it is calculated, the two expressions above are equal. This means that $h \cdot AC = AB \cdot BC$.

Problem 21 *(Student page 43)* Let the lengths of the two parallel sides be a and b, and let the height of the trapezoid be h.



The resulting parallelogram has two sides with length $a + b$, and height $\frac{1}{2}h$. Since the trapezoid has the same area as the parallelogram, the area of the trapezoid can be written as $\frac{1}{2}h(a + b)$.

Problem 22 *(Student page 44)* The formula for the area of a trapezoid is $A = \frac{1}{2}h(a + b)$, where h is the length of the altitude between parallel sides, and a and b are the lengths of these two parallel sides. Suppose the side of length a gets smaller and smaller (approaches 0). Then the area approaches $\frac{1}{2}hb$. In other words, the figure approaches a triangle, so the area approaches the area of a triangle.

One way to show a quadrilateral is a parallelogram is to show that one pair of opposite sides are both congruent and parallel.

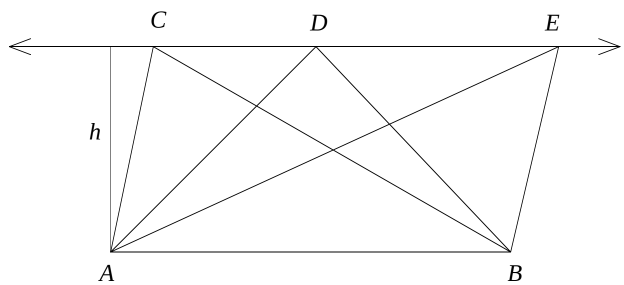
Problem 23 *(Student page 44)* Try drawing a trapezoid with the two parallel sides the same length. You are forced to draw a parallelogram! Notice that in the expression $\frac{1}{2}h(a + b)$, if you let $a = b$, then you get $\frac{1}{2}h \cdot 2a = ha$, which is the area formula for a parallelogram.

Problem 24 *(Student page 44)*

a. When you move one vertex of a triangle, you are keeping the base fixed. If you move the vertex in such a way as to increase the altitude from that vertex to the fixed base, then you will automatically increase the area; similiarly, if you move the vertex in a way which decreases the altitude, then the area will decrease.

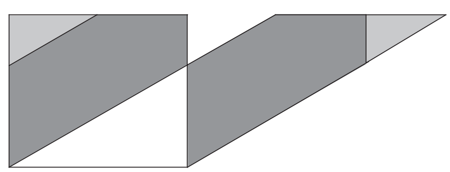
b. If you move the vertex along a line which is parallel to the base, the area of the triangle will not change. As the vertex moves, the height will remain constant, as it will always be equal to the distance between the two parallel lines. The area remains fixed as the vertex moves, as neither the base nor the height is changing.

The picture below illustrates this situation; every triangle with base $\overline{AB}$ and a third vertex on the line $\overleftrightarrow{CDE}$ will have the same area (three such triangles are shown).

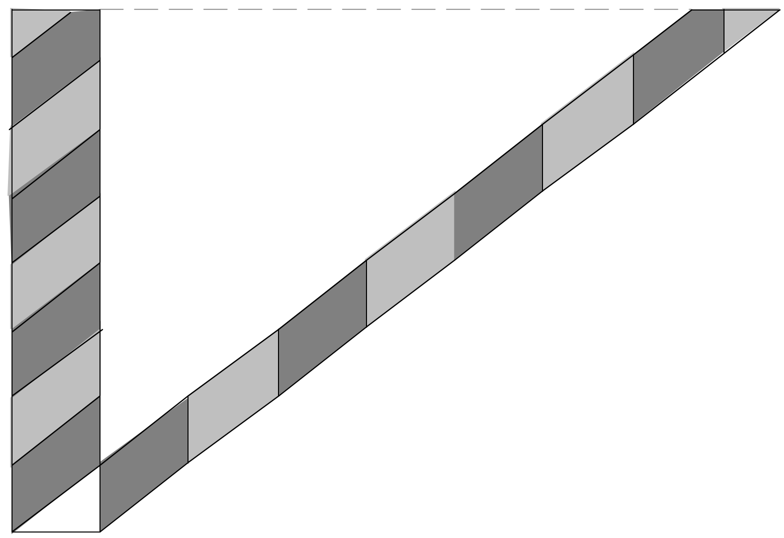


c. The area formula for a triangle, $\frac{1}{2}bh$, shows that the area depends only on the base and height of the triangle, not on the lengths of its sides or its angle measures. Therefore, as long as the base and height remain constant, the quantity $\frac{1}{2}bh$ (the area) will also remain constant.

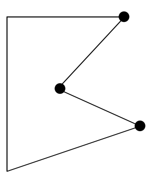
Problem 25 *(Student page 45)* The shading in the picture below shows how to dissect the parallelogram into pieces which fit inside the rectangle.



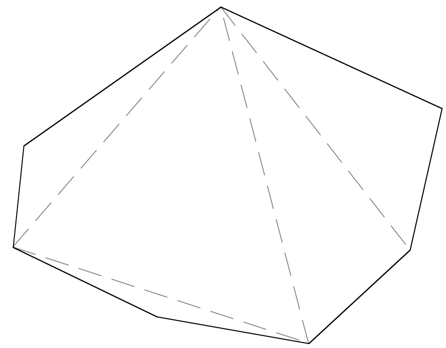
Problem 26 *(Student page 45)* The picture on the next page shows one way to dissect the parallelogram into a rectangle. First, we drew a rectangle with the same base and height as the parallelogram, and then we tried to cut the parallelogram into pieces which filled the rectangle.



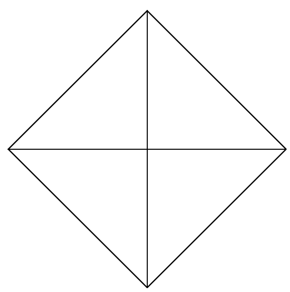
Problem 27 *(Student page 46)* Any polygon can be cut into triangles. The best way to convince yourself of this is to try lots of examples, especially with nonstandard polygons such as the one below. Start with three adjacent vertices in a corner, and draw a line segment which creates a triangle that has these vertices. Now just work with the rest of the polygon; pick three more vertices and form a second triangle. Continue in this manner, "slicing" triangles off of the corners of the polygon, and working with what is left. Eventually, you will have triangulated your polygon. The only time you will run into trouble is if the three vertices you choose form a concavity (see the side picture); in this case you have to pick another three vertices to start with because the triangle formed by these particular three vertices will lie outside of the polygon.



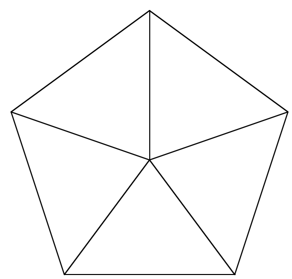
AREA FORMULAS (continued)



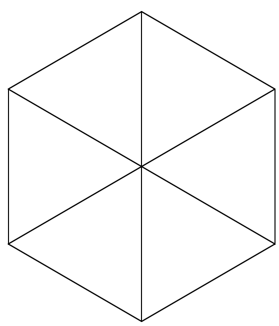
Problems 28–29 *(Student page 46)* The more sides a regular polygon has, the more it looks like a circle. By starting at the center point and drawing segments to the vertices, you can triangulate a regular polygon with n sides into n triangles that are all congruent and hence all have the same area.



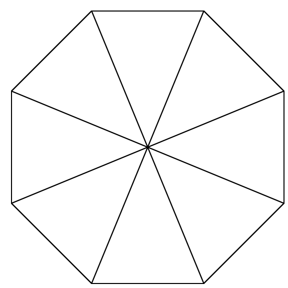
four sides



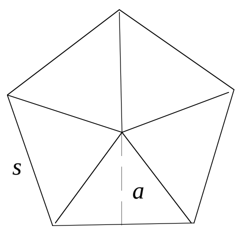
five sides



six sides



eight sides



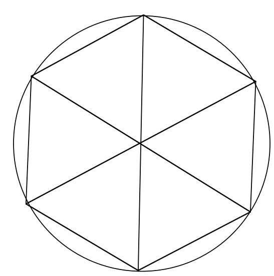
Problem 30 *(Student page 47)* Let a be the distance from the center point of an n-sided polygon to a side of the polygon. It follows that each of the n triangles has an altitude of length a. Let s be the length of each side of the polygon. Then each triangle has an area of $\frac{1}{2}sa$, and since there are n triangles which completely cover the polygon, the total area of the polygon is given by $\frac{1}{2}nsa$.

Or, if you denote the perimeter of the polygon by p, then $p = ns$, since the polygon has n sides, each of length s. Then, the area of the polygon is $\frac{1}{2}pa$.

Problem 31 *(Student page 47)*

a. You can think of the center point of a regular polygon as the point which is equidistant from all the vertices of the polygon. It turns out that every regular polygon can be inscribed in a circle, and the center of this circle is the center point.

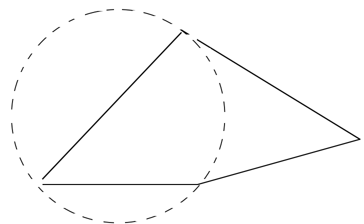
To see this, start with any circle and construct n congruent central angles. This will divide the circle into n arcs, all of equal length. If you connect the endpoints of these arcs, you will have created a regular polygon with n sides.



b. To construct the center point for a regular polygon, construct perpendicular bisectors of two of the sides. Their intersection will be the center point of the polygon.

c. In computing the area of a regular polygon, we assumed that the distance from the center point to each side of the polygon was the same. Notice that all the central angles are congruent, and all the line segments connecting the center point to the vertices of the polygon have the same length, as they are all radii of the circle. By the SAS postulate, all n triangles are congruent, so they must all have the same height. This common height is called the *apothem* of the polygon.

d. There are nonregular polygons that can't be inscribed in a circle, such as the one below. These have no "center point."



Problems 32–33 *(Student pages 47–48)* The figures show that as the number of sides increases, the perimeter of the polygon approaches the circumference of the circle. Also, the apothem approaches the radius of the circle, since, as the number of sides increases, each side gets closer and closer to the circle.

Problem 34 *(Student page 48)* Since the area of the polygon approaches the area of the circle, you can extend your formula for the area of a regular polygon into an area formula for circles.

You proved that the areas of the polygons are given by $\frac{1}{2}pa$, where p is the perimeter of the polygon and a is the apothem. To extend this to circles, replace p by C (the circumference) and replace a by r (the radius). By definition of π, the circumference of the circle satisfies the equation $C = 2\pi r$. Therefore, the area of the circle is given by the expression

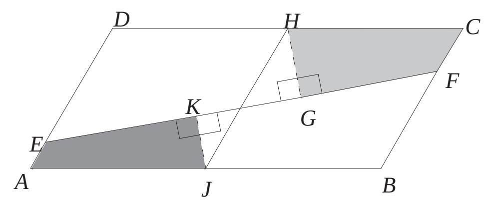
$$\frac{1}{2}Cr = \frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2.$$

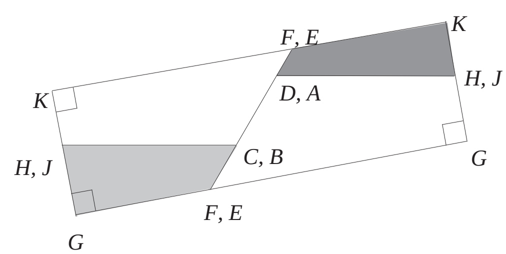
Since this is the same formula we found by approximating a rectangle, we can have more confidence in that argument.

Problem 35 *(Student page 49)* When you cut along segments $\overline{EF}$, $\overline{JK}$, and $\overline{HG}$, you obtain four quadrilaterals, which you need to rearrange to form a rectangle.

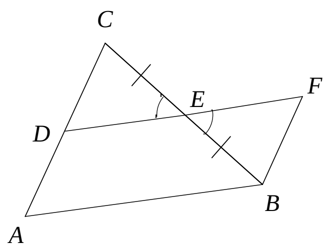
AREA FORMULAS (continued)

In the pictures below, the two smaller quadrilaterals, *AEKJ* and *HCFG* are shaded. The remaining two quadrilaterals are *DHGE* and *KFBJ*. Notice that the base of the resulting rectangle has length *EF* and height *KG*.





More on the Midline Theorem



Problem 2 *(Student page 51)* You are given that $DE = FE$. Since E is a midpoint, you also know that $CE = BE$. $\angle CED \cong \angle BEF$, since they are vertical angles. Applying the SAS postulate, you can conclude that $\triangle DEC \cong \triangle FEB$.

Problem 3 *(Student page 51)* Since $\triangle DEC \cong \triangle FEB$, you can use CPCTC to conclude that $CD = BF$. Since D is the midpoint of $\overline{AC}$, you know that $CD = AD$, and therefore that $AD = BF$.

Apply CPCTC again to see that $\angle DCE \cong \angle FBE$. These two angles are a pair of alternate interior angles, formed by the transversal $\overline{BC}$ to $\overline{AC}$ and $\overline{BF}$. Therefore, $\overline{AC}$ and $\overline{BF}$ are parallel, which means that $\overline{AD}$ is parallel to $\overline{BF}$. Since $ABFD$ has two sides that are both congruent and parallel, it is a parallelogram.

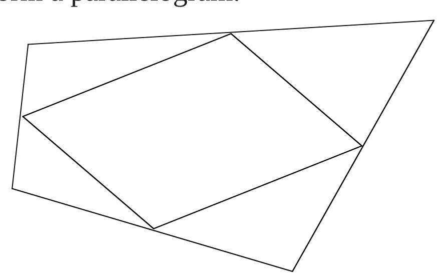
Problem 4 *(Student page 52)* Since $ABFD$ is a parallelogram, $\overline{DF}$ is parallel to $\overline{AB}$, and thus segments $\overline{DE}$ and $\overline{AB}$ are parallel, giving the first part of the Midline Theorem.

Since opposite sides of a parallelogram are congruent, you know that $DF = AB$. By construction, $DE = \frac{1}{2}DF$. Putting these together, you see that $DE = \frac{1}{2}AB$, which proves the second part of the Midline Theorem.

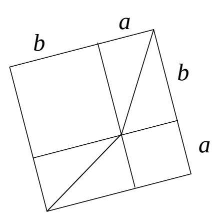
Problem 5 *(Student page 52)* Put the last three problems together to prove the Midline Theorem.

Problem 6 *(Student page 52)* If you define midlines as segments connecting adjacent midpoints of a quadrilateral, here are some things to notice:

- Midlines are parallel to diagonals and half as long.
- Opposite midlines are parallel and congruent to each other.
- All four midlines form a parallelogram.



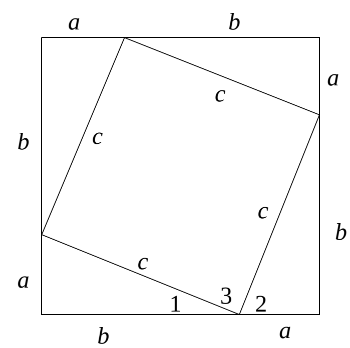
THE PYTHAGOREAN THEOREM



Problem 1 *(Student page 56)* First, check that each of the four small triangles is a right triangle. We know this is the case because each triangle contains one of the right angles of one of the rectangles. Second, notice that each triangle has legs of lengths a and b. By the SAS postulate, all four right triangles are congruent to each other and to the original triangle.

Problem 2 *(Student page 56)* Because each small triangle is congruent to the original, each hypotenuse has length c. Thus, the center figure is a rhombus, with all sides having length c. Now we need to show that the rhombus is actually a square.

Consider the figure below. An angle that is congruent to the smallest angle in the original triangle has been numbered 1, and an angle that is congruent to the middle-sized angle in the original triangle has been numbered 2. Further, one of the angles of the center figure has been numbered 3.



From our original triangle, we know that $m\angle 1 + m\angle 2 + 90^\circ = 180^\circ$. Notice also that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$, since these three angles form a straight line. Putting these two equations together, you see that $m\angle 3 = 90^\circ$, and thus $\angle 3$ a right angle. We can repeat this argument for each angle of the center figure. All of its angles are right angles, so this rhombus must be a square.

Problem 4 *(Student page 59)*

a.–b. For a right triangle with legs of length 3 and 4 inches, the area of the triangle is $\frac{1}{2} \times 3$ inches $\times\, 4$ inches $= 6$ in^2, and a square with sides of length $3 + 4$, or 7 inches, has an area of 49 in^2.

c. The four triangles together have area 24 in^2, so the center square has an area of 49 in$^2 - 24$ in$^2 = 25$ in^2.

d. The Pythagorean Theorem tells us this should be so because $3^2 + 4^2 = 9 + 16 = 25$.

THE PYTHAGOREAN THEOREM (continued)

Problem 5 *(Student page 60)* If you repeat Problem 4, starting with a triangle whose legs are 5 cm and 12 cm long, you will see that the big square has area $(5\text{ cm} + 12\text{ cm})^2 = 289\text{ cm}^2$ and each triangle has area $\frac{1}{2} \times 5\text{ cm} \times 12\text{ cm} = 30\text{ cm}^2$. Thus, the area of the middle square is given by $289\text{ cm}^2 - 4 \times 30\text{ cm}^2 = 169\text{ cm}^2$.

By the Pythagorean Theorem, we have $5^2 + 12^2 = 25 + 144 = 169$.

Problem 6 *(Student page 60)* Since the area of the middle square in Problem 4 is 25 in^2, the sidelength of the square must be 5 inches.

Problem 7 *(Student page 60)*

a. From Problem 5, we know that the area of the square drawn on the hypotenuse of a right triangle with sides 5 cm and 12 cm is 169 cm^2.

b. The side of the square must be 13 cm ($13^2 = 169$), so the perimeter is $4 \times 13 = 52$ cm.

Problem 8 *(Student page 60)*

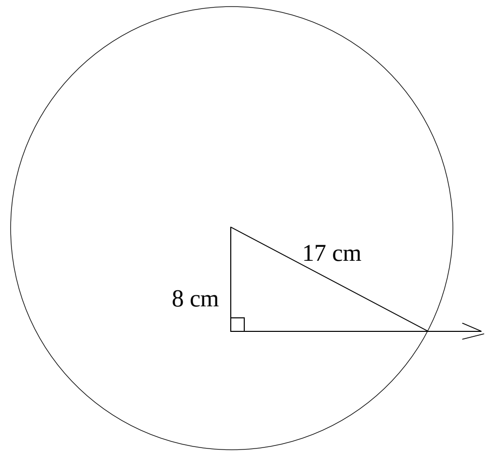
a. Let the remaining leg have length a. Then, the Pythagorean Theorem gives

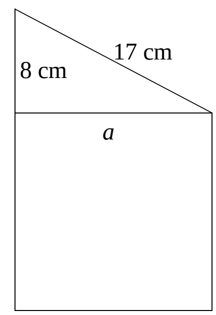
$$\begin{aligned} a^2 + 64 &= 100 \\ a^2 &= 36, \end{aligned}$$

so the remaining square has an area of 36 square feet.

b. The lengths of the three sides are the square roots of the areas: 10 feet, 8 feet, and 6 feet.

Problem 9 *(Student page 60)* To construct the triangle: Draw a right angle with one side fixed at 8 cm and the other side very long. At the endpoint of the 8-cm side (opposite the right angle), construct a circle with that point as a center and a radius of 17 cm. The intersection of that circle with the other side of the right angle gives you the third vertex for your triangle. Finally, you can construct the square on the side of the triangle whose length is unknown.





a. If the length of the missing side is a, we know that

$$\begin{aligned} a^2 + 8^2 &= 17^2 \\ a^2 &= 225, \end{aligned}$$

so the area of the square you have constructed is 225 sq. cm.

b. The side of the square, which forms the other leg of the triangle, will have as its length the square root of this area, or 15 cm.

Problem 10 *(Student page 61)*

a. If you walk along the edge of the parking lot, you will be walking 400 feet (since 100 + 300 = 400).

b. Suppose you walk diagonally, from corner to corner, instead. You can think of this as walking along the hypotenuse of a right triangle with legs of lengths 100 and 300. Using the Pythagorean Theorem, you see that the distance along this diagonal is equal to $\sqrt{100{,}000}$, which means you walk approximately 316 feet. (Use a calculator to approximate the square root; round to the nearest integer.) This is 84 feet shorter than walking along the sides.

c. Remember that the shortest distance between two points is along a straight line, and, in this case, that straight line path is the hypotenuse of the right triangle. If you have to walk in a zigzag path, the distance you walk will increase, since you will no longer be walking along this straight line. However, if you stay close to the diagonal, you will do no worse than traveling along the edges of the lot. And if you can walk any distance actually along the diagonal, the total trip will be shorter.

Problem 11 *(Student page 62)* A square with sides of length one, two, four, ten, and 100 feet will have diagonals with lengths of:

- **a.** $\sqrt{2}$ feet;
- **b.** $2\sqrt{2}$ feet;
- **c.** $4\sqrt{2}$ feet;
- **d.** $10\sqrt{2}$ feet;
- **e.** $100\sqrt{2}$ feet.

Problem 12 *(Student page 62)* In general, if you have a square with sides of length x, and you want to find the length of the diagonal, you can use the Pythagorean Theorem, since the diagonal forms an isosceles right triangle with legs of length x. Thus, if the diagonal has length d, it follows that

$$\begin{aligned} d^2 &= x^2 + x^2 \\ d^2 &= 2x^2 \\ d &= \sqrt{2}x. \end{aligned}$$

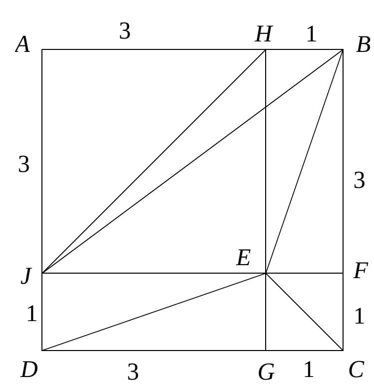
Problem 13 *(Student page 62)* Using the result of Problem 12, the distance from second base to home plate is $90\sqrt{2}$ feet.

Problem 14 *(Student page 62)* It might help to begin by labeling all segments whose lengths you know:

$$CF = HB = DJ = GC = EF = EG = 1$$

and

$$BF = AH = AJ = DG = JE = HE = 3.$$



a. $AB = 4$

b. $\overline{BE}$ is the hypotenuse of right triangle $\triangle BEF$, which has legs of lengths 3 and 1. Thus $(BE)^2 = 9 + 1$, so $BE = \sqrt{10}$.

c. $\overline{BJ}$ is the hypotenuse of right triangle $\triangle BFJ$, which has legs of lengths 3 and 4, so it follows that $BJ = 5$.

d. $\overline{HJ}$ is a diagonal of square $AHEJ$. Since the square has sides of length 3, $HJ = 3\sqrt{2}$.

e. $\overline{CE}$ is a diagonal of square $CGEF$, and this square has sides of length 1. Therefore, $CE = \sqrt{2}$.

f. The area of $AHEJ$ is 9 (a square with sidelength 3).

g. The area of $CGEF$ is 1 (a square with sidelength 1).

h. $\triangle BEF$ is a right triangle with base 1 and height 3, so its area is $\frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}$.

Problem 15 *(Student page 63)*

a. The Pythagorean Theorem says that $1^2 + 3^2 = (EF)^2$, so $EF = \sqrt{10}$.

b. $ABCD$ is a square with sidelength 4, so its perimeter is $4 + 4 + 4 + 4 = 16$.

c. The area of $ABCD$ is $4 \times 4 = 16$.

Problem 16 *(Student page 63)* An equilateral triangle with sides of length one, two, three, ten, or 100 cm will have a height of:

a. $\frac{\sqrt{3}}{2}$ cm;

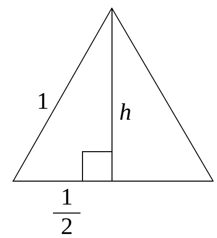
b. $\sqrt{3}$ cm;

c. $\frac{3\sqrt{3}}{2}$ cm;

d. $\frac{10\sqrt{3}}{2} = 5\sqrt{3}$ cm;

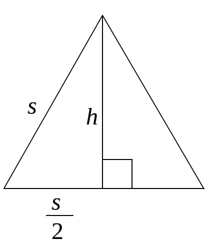
e. $\frac{100\sqrt{3}}{2} = 50\sqrt{3}$ cm.

Problem 17 *(Student page 63)* The answer for Problem 16a can be found by applying the Pythagorean Theorem to half of an equilateral triangle with sidelength 1, which is a right triangle with hypotenuse of length 1. (Recall that every altitude of an equilateral triangle is also a median.)



$$\begin{aligned} a^2 + b^2 &= c^2 \\ \left(\frac{1}{2}\right)^2 + h^2 &= 1^2 \\ \frac{1}{4} + h^2 &= 1 \\ h^2 &= \frac{3}{4} \\ h &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \end{aligned}$$

The other answers are found in the same manner by changing the length of the hypotenuse.



Suppose that you have an equilateral triangle with sides of length s, and you want to find the height, h. In the figure, h is the length of an altitude which forms a right triangle with legs of lengths h and $\frac{s}{2}$ and a hypotenuse of length s. Use the Pythagorean Theorem to see that $h^2 + (\frac{s}{2})^2 = s^2$, so $h^2 = \frac{3s^2}{4}$, implying that $h = \frac{s\sqrt{3}}{2}$.

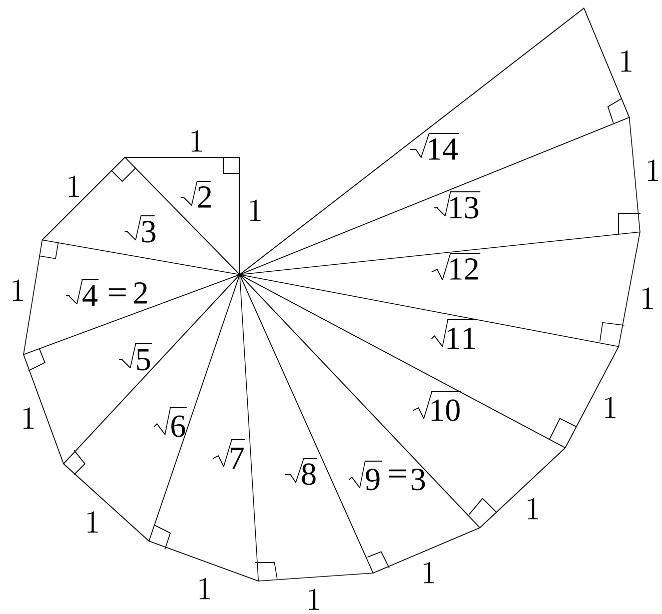
Problem 18 *(Student page 63)* The shortest unlabeled segment has a length of $\sqrt{2}$, since it is the hypotenuse of a right triangle with both legs of length 1.

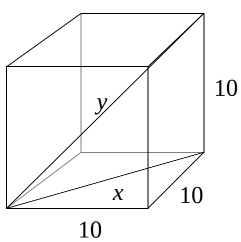
This segment of length $\sqrt{2}$ is one of the legs of the second right triangle; thus, this second triangle has legs of length 1 and $\sqrt{2}$, so its hypotenuse (the second unlabeled segment) has a length of $\sqrt{3}$, since $(\sqrt{2})^2 + 1^2 = (\sqrt{3})^2$.

This segment of length $\sqrt{3}$ is one of the legs of the third right triangle, so this triangle has legs of length 1 and $\sqrt{3}$; hence its hypotenuse (the third unlabeled segment) has length $\sqrt{4} = 2$, since $(\sqrt{3})^2 + 1^2 = (\sqrt{4})^2$.

By now the pattern becomes clear: the nth unlabeled segment has length $\sqrt{n+1}$.

THE PYTHAGOREAN THEOREM (continued)





Problem 19 *(Student page 64)* Picture the cube standing upright, with its base flat. Then, if x is the length of the diagonal of the cube's base, you can find x:

$$\begin{aligned} x^2 &= 10^2 + 10^2 \\ x^2 &= 200 \\ x &= 10\sqrt{2}. \end{aligned}$$

Now the diagonal of the cube can be viewed as the hypotenuse of a right triangle; the legs of this triangle are the diagonal of the base of the cube, and the back vertical edge of the cube. Letting y be the length of the diagonal of the cube, it follows that

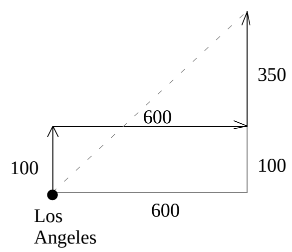
$$\begin{aligned} y^2 &= 10^2 + x^2 \\ y^2 &= 10^2 + (10\sqrt{2})^2 \\ y^2 &= 300 \\ y &= 10\sqrt{3}. \end{aligned}$$

This problem is closely related to Problem 18, although this is not obvious. In Problem 18, if all the labeled segments had length 10, then the nth unlabeled segment would have length $10\sqrt{n+1}$. So imagine that each labeled segment has length 10 inches, and look only at the first two right triangles. You can fold along the segment

THE PYTHAGOREAN THEOREM (continued)

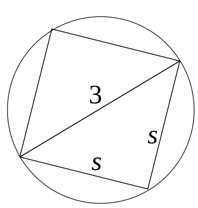
connecting them, to see that the longer of the two unlabeled segments (the hypotenuse of the second triangle) has the same length as the diagonal of a $10 \times 10 \times 10$ inch cube.

To see this, lie the first triangle flat, and fold along its hypotenuse so that the second triangle stands upright. Think of the legs of the first triangle as being two edges of the base of the cube, where the leg of length 2 in the second triangle is an edge of the cube representing its height. Then the hypotenuse of the second triangle plays the role of the diagonal of the cube, and this hypotenuse has length $10\sqrt{3}$ inches.



Problem 20 *(Student page 64)* In this problem, the airplane has traveled north a total distance of 450 miles and has traveled east a total of 600 miles, so calculating the plane's distance from its starting point is the same as finding the length of the hypotenuse of a right triangle with legs of length 450 and 600.

The distance of the plane from Los Angeles is thus given by $\sqrt{(450)^2 + (600)^2}$, which is 750 miles. The reason why this is not the precise distance is that we reduced the problem to two dimensions, but these distances are really on the surface of the Earth, which is a three-dimensional sphere. So to simplify our calculations, we have to sacrifice a bit of accuracy.



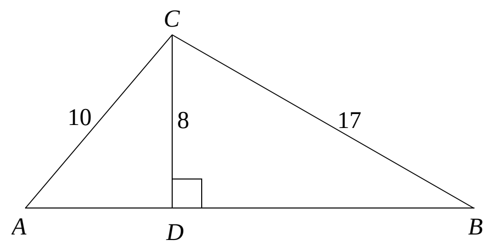
Problem 21 *(Student page 64)* It helps here to realize that a circle has infinitely many diameters. In particular, you can draw the diameter which connects opposite corners of the square.

Then, you see that the 3-inch diameter is really the hypotenuse of a right triangle that has two sides of the square as legs. Let s be the length of the sides of the squares. Apply the Pythagorean Theorem to calculate that

$$\begin{aligned} s^2 + s^2 &= 3^2 \\ 2s^2 &= 9 \\ s^2 &= \frac{9}{2}. \end{aligned}$$

But s^2 is exactly the area of the square, which is what you are asked to find. Hence, the square has an area of $\frac{9}{2}$ square inches.

Problem 22 *(Student page 64)* Triangle $\triangle ABC$ is not a right triangle, but it has been divided into two right triangles. Thus, you can find the area of each right triangle separately, and add these areas together to find the total area of $\triangle ABC$.



Let D be the point where the altitude from C intersects $\overline{AB}$. To find the areas of $\triangle ACD$ and $\triangle DCB$, you first need to find the lengths AD and DB. Applying the Pythagorean Theorem to $\triangle ACD$, you see that

Notice that if you scale the well-known 3–4–5 right triangle by 2, you obtain a 6–8–10 right triangle.

$$AD = \sqrt{10^2 - 8^2} = \sqrt{36} = 6,$$

and

$$DB = \sqrt{17^2 - 8^2} = \sqrt{225} = 15.$$

You can now calculate that

$$\text{Area}(\triangle ACD) = \frac{1}{2} \times 6 \times 8 = 24,$$

and

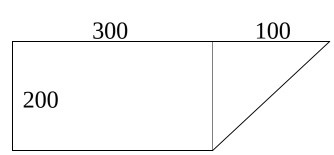
$$\text{Area}(\triangle DCB) = \frac{1}{2} \times 15 \times 8 = 60,$$

so

$$\text{Area}(\triangle ABC) = 24 + 60 = 84.$$

We are assuming that the two angles on the left are, in fact, right angles even though they are not marked that way. Without that assumption, we don't have a way to solve the problem.

Problem 23 *(Student page 64)* Draw a perpendicular from the lower right vertex to the top segment, dividing the plot of land into two pieces: a rectangle and a square. The rectangle has an area of 60,000 square feet, while the triangle has an area of 10,000 square feet. Thus, the area of the entire plot is 70,000 square feet.



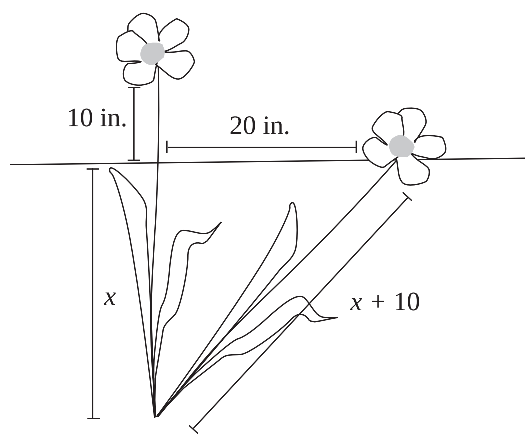
Using the same assumption that the two angles on the left are right angles, and thus the top and bottom segments are parallel, we see that the complete figure is a trapezoid.

Thus, another way to find the area of the plot is to use the area formula for trapezoids:

$$\begin{aligned} A &= \frac{1}{2}h(a+b) \\ &= \frac{1}{2}(200)(400+300) \\ &= 100(700) \\ &= 70{,}000, \end{aligned}$$

so the area of the plot is 70,000 square feet.

Problem 24 *(Student page 65)* Call the depth of the pond x. Then, as you can see from the picture, the length of the stem of the flower is $x + 10$. This length doesn't change as the blossom is pulled to the side.



When the stem is pulled to the side, it becomes the hypotenuse of a right triangle with legs of lengths x and 20. Therefore, you can calculate that

$$\begin{aligned} (x+10)^2 &= x^2 + 20^2 \\ x^2 + 20x + 100 &= x^2 + 400 \\ 20x &= 300 \\ x &= 15. \end{aligned}$$

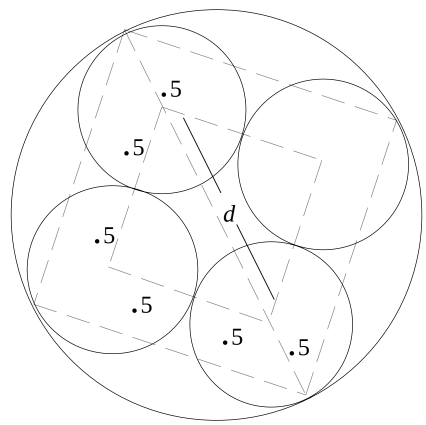
The pond is 15 inches deep.

Problem 26 *(Student page 65)*

a. The diameter of the large circle is the same as the diagonal of the large square. Let d be the length of the diagonal of the small square. Notice that the diagonal of the large square has length

$$0.5 + d + 0.5 = 1 + d,$$

where 0.5 represents the radius of the small circles. We want to find the value of $1 + d$.

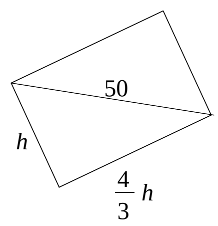


Since d is the length of the diagonal of the small square, and this diagonal can be thought of as the hypotenuse of a right triangle, you can use the Pythagorean Theorem. Each side of the small square has length 1 mm, since it can be broken down into two radii of length .5 mm. Thus, $d = \sqrt{2}$, since $1^2 + 1^2 = d^2$.

Therefore, the diameter of the large circle is $1 + \sqrt{2}$.

b. This is not the same as asking about the cables because the cables are three-dimensional—they have thickness which would have to be taken into consideration.

Problem 27 *(Student page 66)* There are two ways to approach this problem. You know that a right triangle with legs of length 3 and 4 has a hypotenuse of length 5. If you are familiar with similar triangles, then you know that the fact that the ratio of the width to the height of a television screen is 4:3 means that the right triangle formed by the screen's diagonal must be a scaled copy of the 3–4–5 right triangle. If the diameter of the screen is 50 inches, then the width and height of the screen must be 40 inches and 30 inches, respectively, giving a viewing area of 1200 square inches.



What if you don't know about similar triangles? If the width and height of the screen have lengths w and h, then you are told that $\frac{w}{h} = \frac{4}{3}$, so you can calculate that $w = \frac{4}{3}h$. If the screen has a diameter of 50 inches, then

$$\begin{aligned} w^2 + h^2 &= 50^2 \\ \left(\frac{4}{3}h\right)^2 + h^2 &= 50^2 \\ \frac{16}{9}h^2 + h^2 &= 50^2 \\ \frac{25}{9}h^2 &= 50^2 \\ \frac{5}{3}h &= 50 \\ h &= 30. \end{aligned}$$

Since $w = \frac{4}{3}h$, it follows that $w = 40$, and $A = hw = 40 \times 30 = 1200$ square inches.

Problem 28 *(Student page 66)* Two examples of Pythagorean triples are (5,12,13) and (8, 15, 17). It's important to note that Pythagorean triples must contain only positive *integers*, so $(1, 1, \sqrt{2})$ is not a Pythagorean triple.

Problem 29 *(Student page 66)* Each triple given can be obtained from the (3, 4, 5) triple by multiplying each integer by a fixed amount: 1, 2, 10, 15, and 100. This is why they are considered a family. These triples are actually part of a larger family; the family consisting of *all* integral multiples of (3, 4, 5). This family consists of all Pythagorean triples of the form

$$(3t, 4t, 5t)$$

where t can be any positive integer.

Geometrically, the triangles obtained from these triples will all be scaled copies of the 3–4–5 right triangle.

Problem 30 *(Student page 66)* The three triangles you draw will all be right triangles and will be scaled copies of each other.

Problem 31 *(Student page 67)* The (5, 12, 13) Pythagorean triple is not part of the (3, 4, 5) family. To see this, you can try to solve the following equations:

$$\begin{aligned} 5 &= 3t \\ 12 &= 4t \\ 13 &= 5t. \end{aligned}$$

Because each equation has a different solution, the two triples are not in the same family. Here are some other members of the (5, 12, 13) family:

- (10, 24, 26)
- (50, 120, 130)
- (35, 84, 91)
- (125, 300, 325).

One method of finding these equations comes from coordinate geometry, and another comes from the Gaussian integers. See the "Mathematics Connections" section for this investigation in the *Teaching Notes* for a full discussion of this problem.

Problem 32 *(Student page 67)* This challenge problem requires a lot of algebra. It turns out that if r and s are any positive integers such that r is larger than s and they share no common factors, then

$$\begin{aligned} x &= r^2 - s^2 \\ y &= 2rs \end{aligned}$$

and

$$z = r^2 + s^2$$

form a primitive Pythagorean triple. (A triple is *primitive* if x, y, and z share no common factors.)

We can verify algebraically that in general three integers generated by the three given equations always form a Pythagorean triple.

$$\begin{aligned} x^2 &= (r^2 - s^2)^2 = r^4 - 2r^2s^2 + s^4 \\ y^2 &= (2rs)^2 = 4r^2s^2 \\ z^2 &= (r^2 + s^2)^2 = r^4 + 2r^2s^2 + s^4 \end{aligned}$$

Then,

$$\begin{aligned} x^2 + y^2 &= (r^4 - 2r^2s^2 + s^4) + (4r^2s^2) \\ &= r^4 + 2r^2s^2 + s^4 \\ &= z^2. \end{aligned}$$

Problems 33–34 *(Student page 67)* Suppose you are given a triangle with sides of length a, b, and c, such that $a^2 + b^2 = c^2$. You want to show that the triangle is a right triangle.

Construct a right triangle with legs of lengths a and b. By the Pythagorean Theorem, the hypotenuse must have length c, since you know that c is the number satisfying the equation $a^2 + b^2 = c^2$. So now you have two triangles with sides of length a, b, and c: the triangle you started with, and the right triangle you constructed. But the SSS congruence postulate states that these two triangles must be congruent. Therefore, the original triangle must be a right triangle.

THE PYTHAGOREAN THEOREM (continued)

Problem 35 *(Student page 68)* Point A has coordinates $(4, 3)$. The right triangles have legs of lengths 3 and 4, and hypotenuse of length 5. Therefore, the distance from A to the origin, which is the length of the hypotenuse OA, is 5.

Problem 36 *(Student page 68)* In this problem, the distances from points A, B, C, D, and E to the origin are $\sqrt{13}$, $\sqrt{2}$, $\sqrt{13}$, $\sqrt{5}$, and $\sqrt{8}$, respectively.

The key fact here is that if a point has coordinates (a, b), then you can form a right triangle that has legs of lengths $|a|$ and $|b|$, and the length of the hypotenuse is equal to the distance from the point to the origin. Thus, the distance from point (a, b) to the origin is $\sqrt{a^2 + b^2}$.

Problem 37 *(Student page 69)* In the given picture, $C = (3, -1)$, and $D = (1, 1)$. Both legs of the right triangle formed by these points have length 2. Thus, the distance between C and D is $\sqrt{8}$. Notice that

$$\sqrt{(3-1)^2 + (-1-1)^2} = \sqrt{8}.$$

Problem 38 *(Student page 69)* In this case, $E = (-2, 2)$ and $F = (-1, -1)$. The right triangle formed has legs of lengths 1 and 3, and so the distance between E and F is equal to $\sqrt{10}$. Notice that

$$\sqrt{[-2-(-1)]^2 + [2-(-1)]^2} = \sqrt{10}.$$

Problem 39 *(Student page 69)* The distance between

$$G = (x_1, y_1)$$

and

$$H = (x_2, y_2)$$

is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Notice that, because each term inside the square root is squared, it does matter if either the difference between the x-coordinates or the difference between the y-coordinates is negative.

Problem 40 *(Student page 70)* If you project A to the xy-plane, you get the point $(3, 6)$, which is $\sqrt{45}$ units away from the origin, since

$$3^2 + 6^2 = 45^2.$$

THE PYTHAGOREAN THEOREM (continued)

In three dimensions, you can form a right triangle with legs of lengths $\sqrt{45}$ and 2, and the hypotenuse of this triangle is the three-dimensional distance from A to the origin. Therefore, this distance is equal to 7, since

$$(\sqrt{45})^2 + 2^2 = 7^2.$$

Notice that

$$7 = \sqrt{3^2 + 6^2 + 2^2},$$

so this is a generalization of the formula for two dimensions.

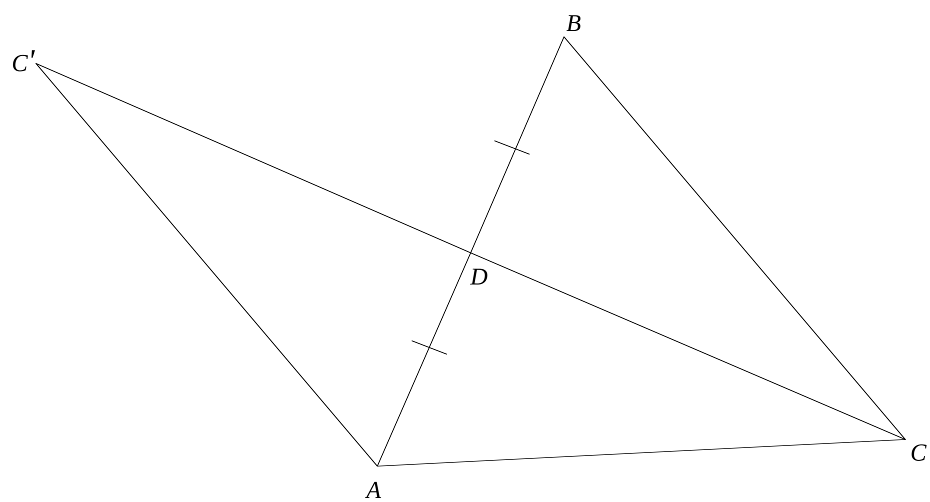
Problem 41 *(Student page 70)* In three dimensions, the distance from a point (x, y, z) to the origin is given by $\sqrt{x^2 + y^2 + z^2}$.

The distance between two points is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

CHANGING SHAPE

Problem 1 *(Student page 71)* Suppose that $\angle ACB$ is the smallest angle of $\triangle ABC$. Let D be the midpoint of side $\overline{AB}$, and cut along median $\overline{CD}$. Rotate $\triangle CBD$ about D 180°, aligning segments $\overline{AD}$ and $\overline{BD}$. Call the image of point C under this rotation C'; consider triangle $\triangle AC'C$. (You know this is a triangle, since the two halves of $\overline{AB}$ lined up exactly.) This new triangle clearly has the same area as the original; it remains to show that none of the angles in $\triangle AC'C$ are congruent to any of the angles in $\triangle ABC$.



Notice that $m\angle ACD < m\angle ACB$, and so $m\angle ACD$ is smaller than each angle of $\triangle ABC$, as $\angle ACB$ is the smallest angle in the triangle.

Similarly, $m\angle DC'A < m\angle ACB$, because $m\angle DC'A = m\angle DCB < m\angle ACB$. Thus, $m\angle DC'A$ is also less than each angle in $\triangle ABC$.

Finally, consider $\angle C'AC$. Notice that it is composed of the two largest angles from the original triangle, $\angle ABC$ and $\angle BAC$. Thus, $\angle C'AC$ is larger than either $\angle ABC$ or $\angle BAC$, and is therefore also larger than the measure of the smallest angle, $\angle ACB$.

So all three angles in $\triangle C'AC$ are different from the angles in $\triangle ABC$.

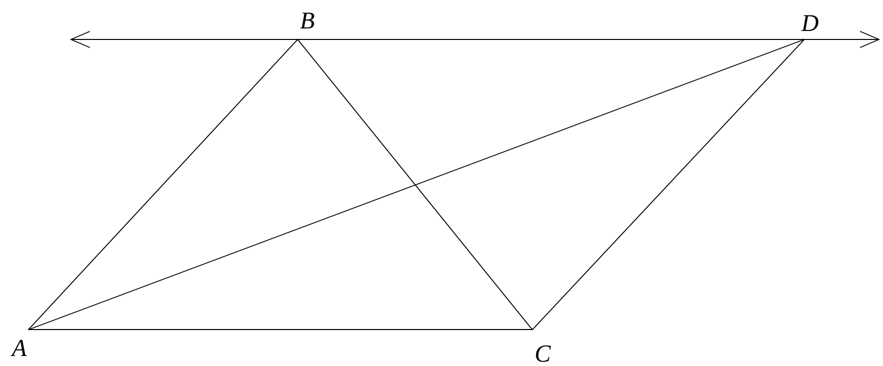
To summarize, $\triangle AC'C$ is obtained by rearranging $\triangle ABC$, and yet has no angles congruent to any of the angles of $\triangle ABC$.

Problem 2 *(Student page 72)* Here is one way to construct another triangle with the same area as a given triangle, $\triangle ABC$, but with no angles congruent to those of

CHANGING SHAPE (continued)

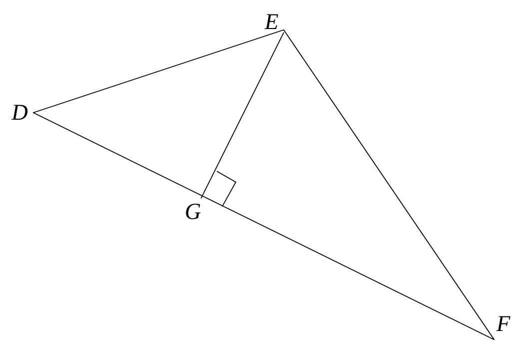
$\triangle ABC$. It relies on the fact that any two triangles which share the same base and height will have the same area.

Draw a line through B parallel to $\overline{AC}$, and construct a point D on this line. As you drag D along the line, you be will forming a family of triangles, all having the same area as $\triangle ABC$; this is because for any location of D, triangles $\triangle ABC$ and $\triangle ADC$ share base $\overline{AC}$, and both have height equal to the distance between the two parallel lines. Now you just have to find a location for D satisfying the property that no angle in $\triangle ADC$ is congruent to any angle in $\triangle ABC$.



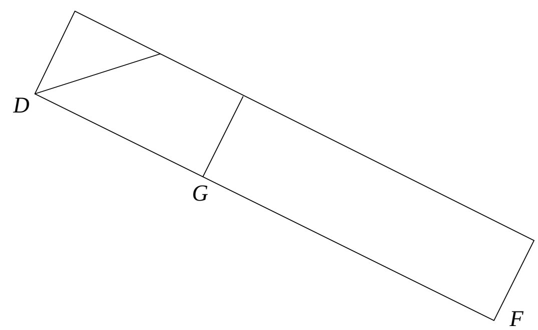
The plan is to construct a triangle with one side congruent to $\overline{CD}$ and two other sides longer than $\overline{AB}$, so all three sides will be different from the original triangle.

Problem 3 *(Student page 72)* Suppose that side $\overline{DF}$ is the longest side of the given triangle. Notice that the altitude to $\overline{DF}$ ($\overline{EG}$ in the picture below) is shorter than any side of $\triangle DEF$.

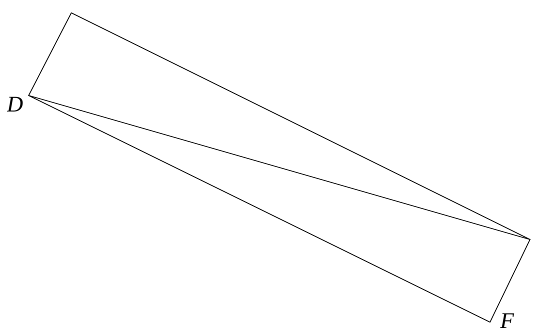


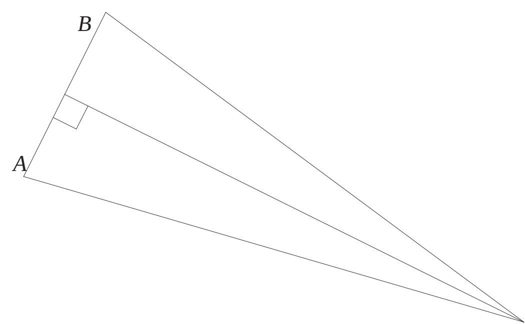
CHANGING SHAPE *(continued)*

First, dissect the triangle into a rectangle with base $\overline{DF}$ and height $\frac{1}{2}EC$, using any of the algorithms you created earlier.



Then, cut along a diagonal of the rectangle, and turn it into an isosceles triangle with base twice the smaller side of the rectangle.

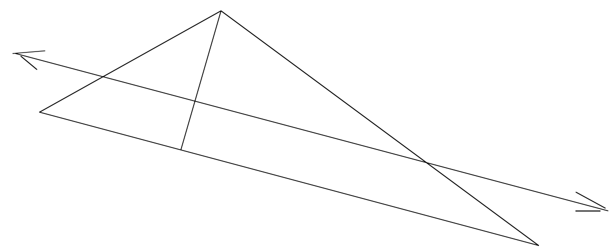




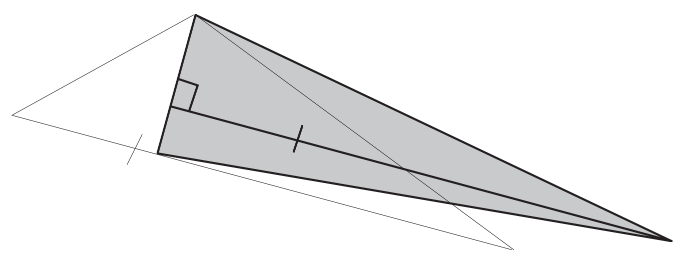
Since the smallest side of the new triangle is twice the smaller side of the rectangle, it is congruent to the original triangle's altitude, $\overline{EG}$, which was shorter than any side of the original triangle. Notice that the altitude of the new isosceles triangle is the same as the longer side of the rectangle, which was the same as the longest side of our original triangle, $\overline{DF}$. The two congruent sides are both longer than the altitude between them, so they are longer than all three sides of the original triangle.

CHANGING SHAPE (continued)

Problem 4 *(Student page 72)* A similar strategy to the one used in Problem 3 will work here. Using either geometry software or paper and pencil, you can create the altitude to the longest side of a triangle, which will be shorter than all three sides. Then create a perpendicular to the altitude anywhere but at the endpoints.



From the intersection of this perpendicular with your altitude, mark off a length equal to the length of the longest side of your original triangle. Now connect the other endpoint of this segment with the two endpoints of your altitude.



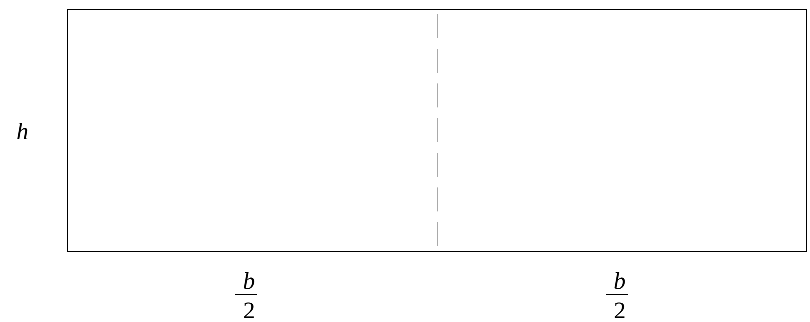
The conclusions here depend on the fact that the perpendicular to the altitude was not drawn at an endpoint. Why?

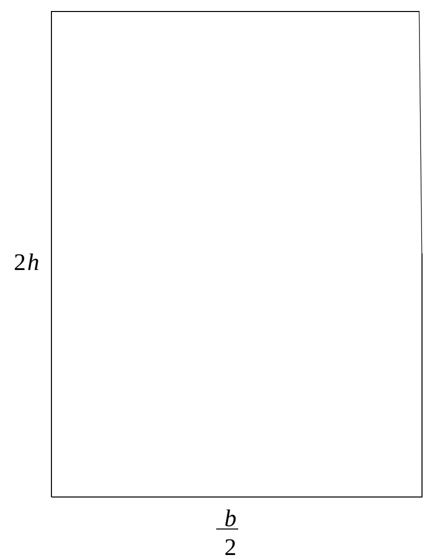
The areas of the two triangles are equal. If the first has longest side b and height to that side h, the second has shortest side h and height to that side b. Both triangles have area $\frac{1}{2}bh$.

None of the sides are congruent: The shortest side of the new triangle is shorter than any side of the original triangle, and the other two sides of the new triangle are both longer than the longest side of the original.

A similar method will still work if you start with a rectangle having the property $b = 2h$; you just have to cut perpendicular to the height, not the base.

Problem 5 *(Student page 73)* You can check that the given rectangle has the property that its base, b is greater than twice its height, h. Connect the midpoints of the two base sides of the rectangle, and cut along this segment. This will form two smaller rectangles with base $\frac{b}{2}$ and height h. Now stack one on top of the other, forming a rectangle with base $\frac{b}{2}$ and height $2h$.

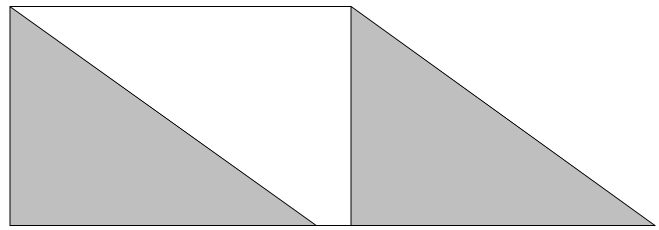


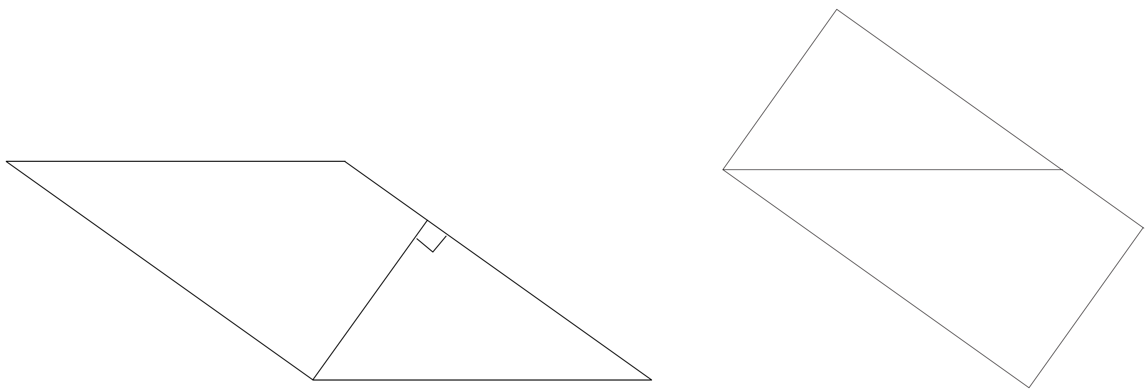


This new rectangle has the same area as the original, and it also has no sides congruent to the sides of the original. The only way the dimensions could be the same would be if $b = 2h$, since you know that $2h$ is definitely not equal to h. But we started by checking that $b > 2h$.

CHANGING SHAPE (continued)

Problem 6 *(Student page 75)* The pictures below demonstrate the algorithm.





Problem 7 *(Student page 75)* To begin, position your rectangle so that the base is longer than the height. The new base is too short to reach the base of the original rectangle. Connect the midpoints of the two heights of the original rectangle.

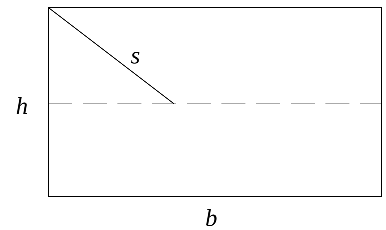
CHANGING SHAPE (continued)

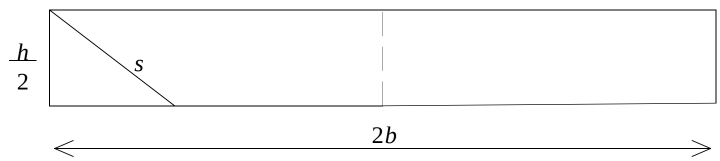
The point of this problem is that your starting rectangle must have height $\leq$ the given side of your new rectangle. If it doesn't, you need to create one that does.

In other words, if the original rectangle had sides of length *b* and *h*, you just transformed it into a rectangle with sides of length 2*b* and $\frac{h}{2}$.

Since the new base length is too short, you are simply shortening the rectangle.

Let s be the segment representing the new base. Since s can be positioned so that it touches the segment you've just drawn, cut along this segment (forming two congruent rectangles) and slide one of them next to the other. s now touches the base of the new rectangle, so you can proceed with the algorithm.



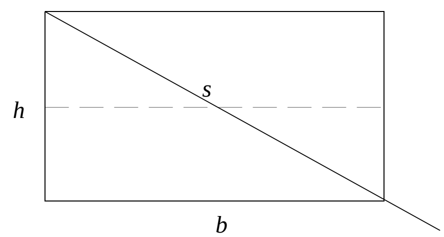


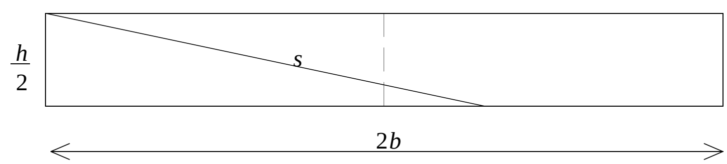
What if s is so short that it doesn't even touch the segment joining opposite midpoints? Then you need to create a rectangle from your original one with height $\leq s$. The process above cuts the height in half, so it can be repeated as many times as necessary, until $\frac{h}{2^n} \leq s$.

Problem 8 *(Student page 75)* Once again, start with your rectangle positioned so that its base is longer than its height. If the desired base, s, is too long to touch the base of your rectangle, then perform the same dissection as in Problem 7. This has

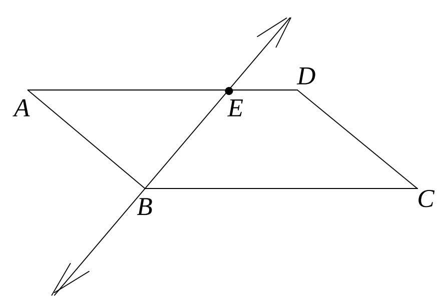
The point of *this* problem is that your starting rectangle must have a base $\geq$ the given side of your new rectangle. If it doesn't, you need to create one that does. Luckily, the same method will work!

the effect of creating a new rectangle with the same area as the original, but with a longer base. Keep doing this until you obtain a rectangle whose base is long enough so that s can be positioned correctly.

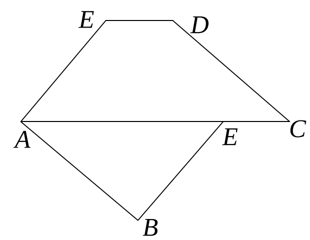




Problem 9 *(Student page 76)* Suppose the perpendicular in Step 4 falls outside the parallelogram. Label the vertices of the parallelogram A, B, C, and D. Let $\overline{AB}$ be the side of the parallelogram whose length is the desired base, and let E be the point where the perpendicular from B to $\overleftrightarrow{CD}$ (the line containing side $\overline{CD}$) intersects $\overline{AD}$.

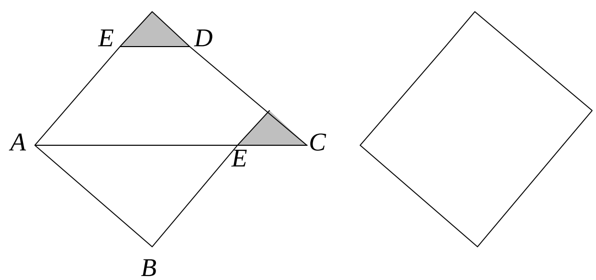


Make the cut along $\overline{BE}$, creating a trapezoid ($BEDC$) and a triangle ($\triangle ABE$). Slide $BEDC$ "northwest," aligning $\overline{BC}$ with $\overline{AE}$. ($\overline{BC}$ will be longer; be sure to match up A and B.)

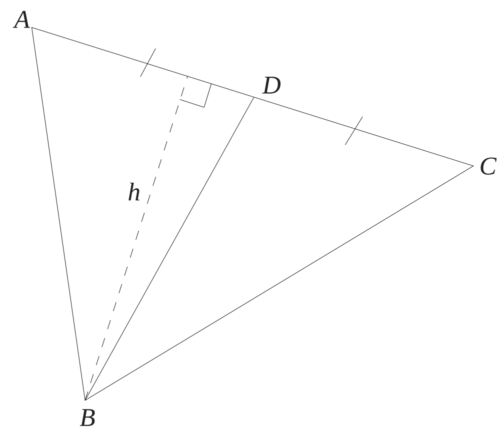


Why does the small triangle fit?

Extend $\overline{BE}$ to make a perpendicular with $\overline{DC}$, forming a small right triangle. Cut off the triangle, and slide it "northwest," aligning $\overline{EC}$ with $\overline{ED}$. You have now formed a rectangle with base $\overline{AB}$.

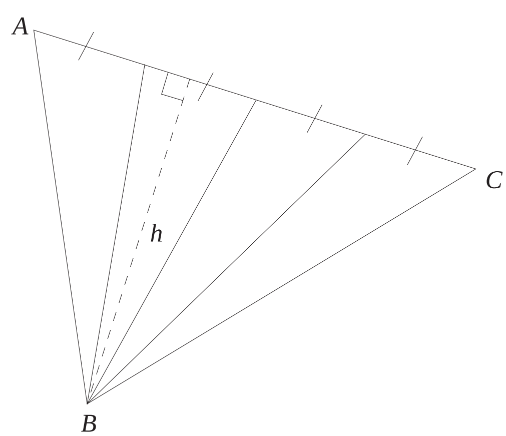


Problem 12 *(Student page 77)* Let D be the midpoint of $\overline{AC}$. Cut along the median $\overline{BD}$ to form two triangles of equal area (actually, you can cut along any median). The two resulting triangles will each have a base of length $\frac{AC}{2}$, and they both will have the corresponding height equal to the perpendicular distance from B to $\overline{AC}$. Thus, they will have the same area.



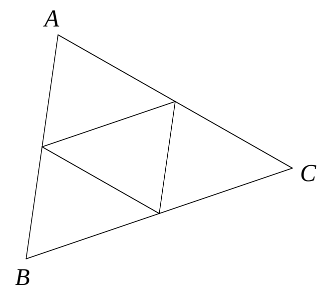
△ *ABD* and △ *BDC* have the same area.

Problems 13–14 *(Student page 77)* One way to divide $\triangle ABC$ into four triangles of equal area is to generalize the method from Problem 12. Divide $\overline{AC}$ into four congruent pieces, and draw three segments from B to $\overline{AC}$, forming four triangles with the same length bases ($\frac{AC}{4}$). As above, they will all have height equal to the perpendicular distance from B to $\overline{AC}$, so they will all have the same area.

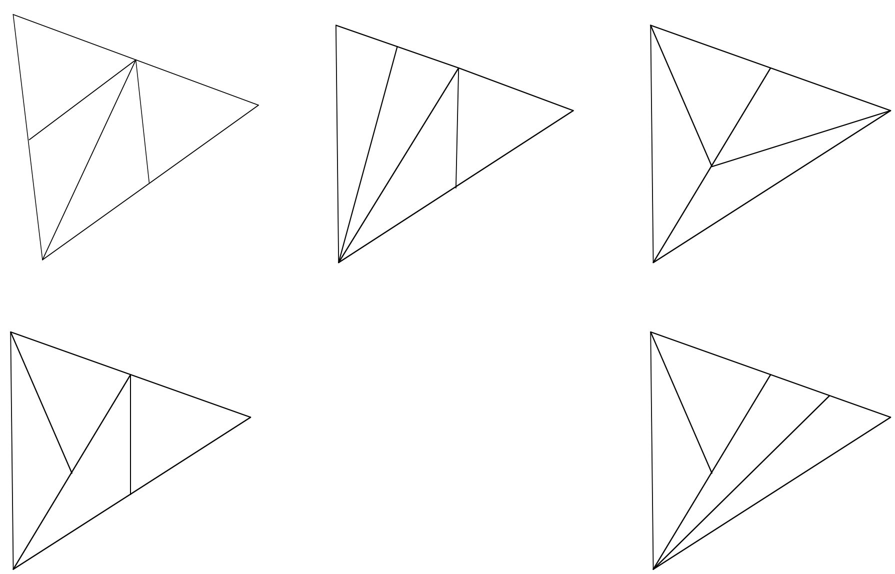


Remember the Midline Theorem, which states that each midline of a triangle is half the length of the third side. Using this, you can see that each of the four small triangles has sides of length $\frac{AB}{2}$, $\frac{BC}{2}$, and $\frac{AC}{2}$, and so the four small triangles are congruent by SSS.

Another way to construct four triangles of equal area is to connect midpoints of consecutive sides of $\triangle ABC$. All four triangles will be congruent, and hence will all have the same area.



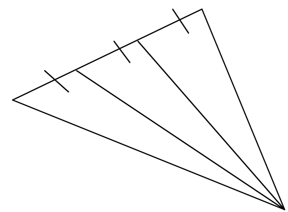
For yet another method, first divide $\triangle ABC$ into two triangles of equal area, as in Problem 12, by drawing the median from any vertex. Now divide each of these triangles into two triangles of the same area (by drawing a median in each); this will divide $\triangle ABC$ into four equal-area triangles. Because of the choices involved in deciding from which vertices to draw medians, this method produces numerous ways to divide $\triangle ABC$ into four equal-area triangles, some of which are shown below.



Problem 15 *(Student page 77)*

One method is to proceed as in Problem 12, only this time draw two segments from

one vertex to the opposite base, dividing the base into three congruent pieces, creating three triangles with equal area.

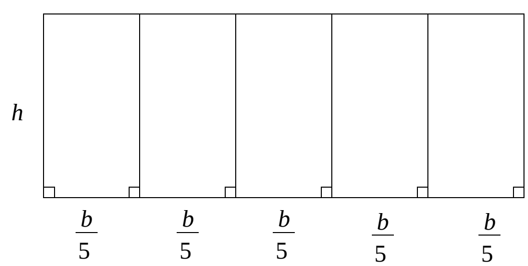


Problem 16 *(Student page 78)*

a. To divide a rectangle into two equal-area rectangles, find the midpoints of two opposite sides and connect them. You may also find the midpoint of one side and draw a perpendicular segment from that point to the opposite side.

b. To divide a rectangle into four equal-area rectangles, take the two equal area rectangles from part a and repeat the process to divide each of them into two equal-area rectangles.

c. To divide a rectangle into five rectangles having equal area, just divide the base into five congruent pieces, and draw four perpendicular lines. If the original rectangle had base b and height h, this will create five new rectangles, all with base $\frac{b}{5}$ and height h; thus they all have the same area.



Problem 17 *(Student page 78)*

The method illustrated in Problem 16 works to produce any number of equal-area rectangles, so there is no number of pieces that is not possible (although, eventually, each piece will become too small to draw by hand).

Problem 18 *(Student page 78)*

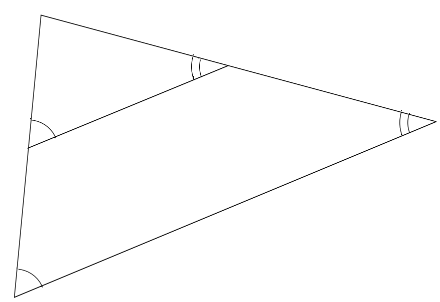
a. Cutting along a median always produces two triangles of equal area, as in Problem 12.

b. Cutting along an altitude will produce two triangles of equal area only when the original triangle is isosceles. In this case, the altitude will also be a median, and the two triangles produced will be congruent.

And, unless the triangle is equilateral, only one of the altitudes and one of the angle bisectors will work.

c. Again, this only works for an isosceles triangle, because in an isosceles triangle, an angle bisector is also a median.

d. True; if two triangles have the same angles and the same area, they are congruent. In general, if two triangles have the same angles you can fit one inside the other by lining up two corresponding angles, and moreover, the opposite sides will be parallel, like this:



Two triangles with the same angles

However, if you also know that the triangles have the same area, then the bases must coincide, and the triangles must actually be congruent.

In the language of similarity, if two triangles have the same angles they are similar, and similar triangles with the same area are congruent.

e. If two triangles have the same sidelengths, they are congruent by SSS; hence they have the same area.

f. No, two triangles can have the same area, but different sidelengths. Recall, for example, Problem 3.

g. Again, see Problem 3 for an example of two triangles with differing side-lengths but the same area.

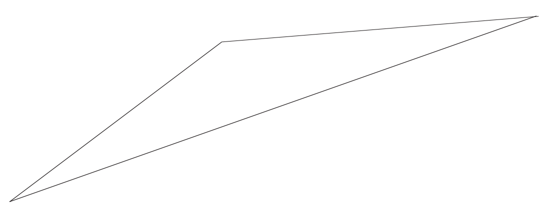
h. See the figure for part d of this problem for an example of two triangles with congruent angles but different areas.

Problem 19 *(Student page 79)* Here are two things you might notice (there are others):

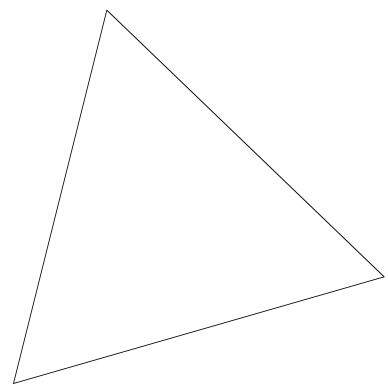
- You use the method for dissecting a parallelogram into a rectangle as part of the rectangle-to-rectangle algorithm.
- The method for parallelogram to rectangle doesn't allow you to specify a base, but the rectangle-to-rectangle algorithm does.

Problem 20 *(Student page 79)*

a. False; look at the two triangles below. Each side in the first triangle is longer than any side in the second, but the second has greater area. Notice that the second one has a much greater height; triangle area, in part, depends on height.



(All three sides are greater than 1.)
Base $= 2$, *other sides* $= \frac{\sqrt{17}}{4}$
Area $= \frac{1}{4} = 0.25$



All sides $= 1$
Area $= \frac{\sqrt{3}}{4}$
(about 0.43)

CHANGING SHAPE *(continued)*

b. True; suppose rectangle A has sidelengths a_1 and a_2, while rectangle B has sidelengths b_1 and b_2. If $b_1 \geq a_1$ and a_2 and $b_2 \geq a_1$ and a_2, then it certainly follows that $b_1 b_2 \geq a_1 a_2$. So rectangle B has greater area.

c. True; the figures above show that sidelengths of triangles may increase, while the area decreases.

Think about starting with an equilateral triangle and "squishing" it to make it long and skinny. As long as the height shrinks faster than the base grows, the area will decrease even though the sides are getting longer.

d. False; suppose a rectangle has sides of length b_1 and b_2. Its area is then $b_1 b_2$, and this value cannot increase if both b_1 and b_2 decrease.

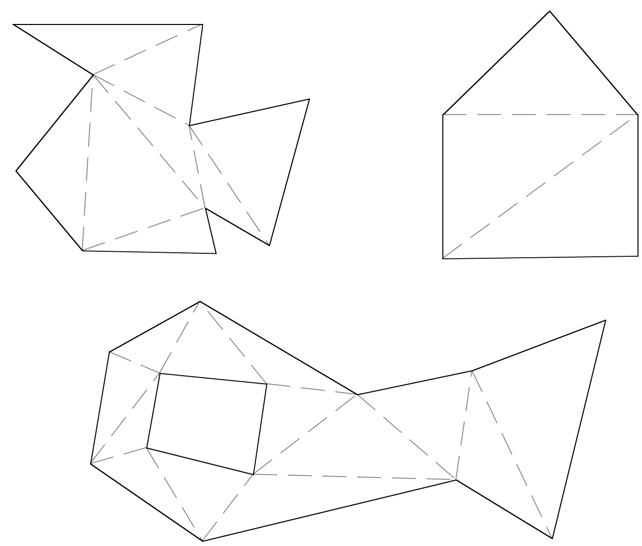
Problem 21 *(Student page 79)* Suppose that you have two rectangles, one with sides of lengths a and b, the other with sides of lengths c and d, and that these rectangles have the same area: $ab = cd$. Suppose further that you want to transform the first rectangle into the second. Use the algorithm in this investigation to transform the $a \times b$ rectangle into a rectangle with base c. Call the height of this new rectangle h. The new rectangle has the same area as the $a \times b$ rectangle, so it follows that $ch = ab$, so in fact, $ch = cd$, implying that $h = d$. Therefore, you've obtained the $c \times d$ rectangle.

EQUIDECOMPOSABLE FIGURES

Problem 1 *(Student page 80)* If you have a rectangle and a square with the same area, you can decompose one onto the other. Suppose the sides of the rectangle have lengths b and h, while the sides of the square have length s. Use the algorithm from the previous investigation to transform the rectangle into another rectangle with side s. Since area is preserved, the new rectangle has area bh, with one side of length s. Since $bh = s^2$, it turns out that all sides of the new rectangle must have length s, so you have transformed the rectangle into the square.

You can't cut a circle to fit inside the triangle. You will learn in this investigation, though, that any two rectilinear figures with the same area are equidecomposable.

Problem 2 *(Student page 82)* Here's one way to triangulate each figure, although there are many other correct triangulations.

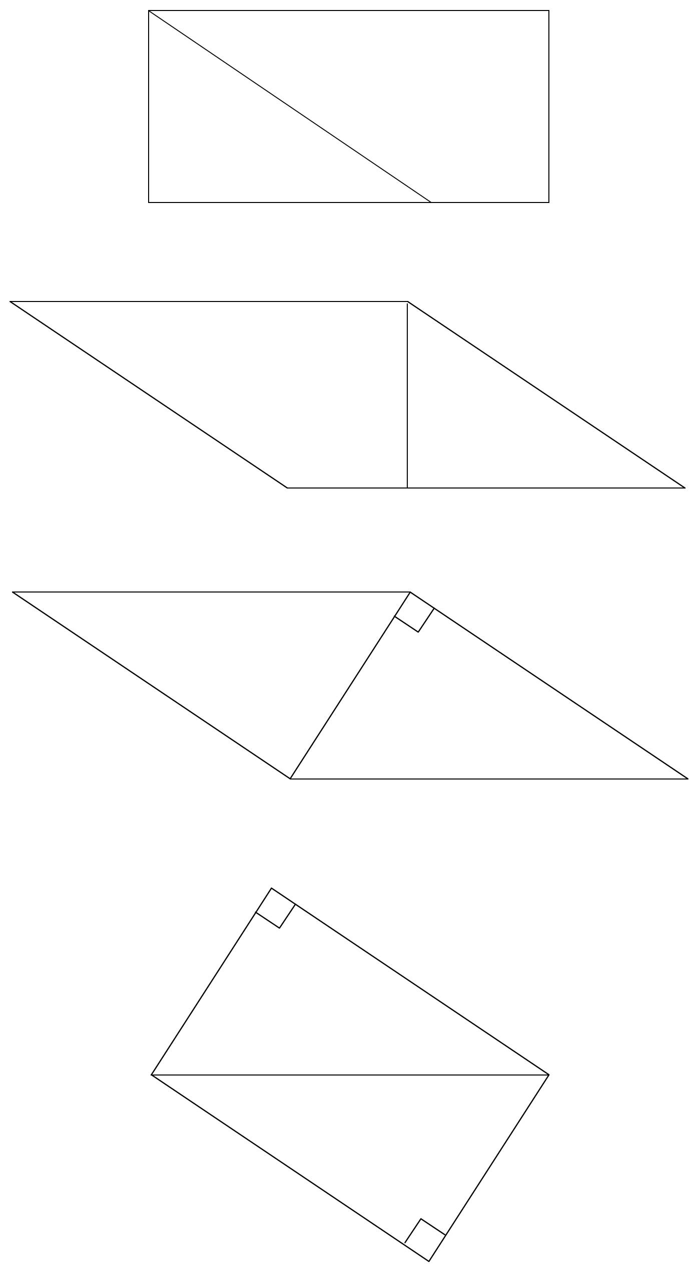


Problem 3 *(Student page 82)* When looking for counterexamples, make sure to try some really oddly-shaped figures: ones with lots of sides, many holes, and lots of concavities, for instance. If the lemma is true, however, you won't be able to find one that can't be cut into triangles.

Problem 4 *(Student page 82)* It is really difficult to write a foolproof algorithm that will work to triangulate *any* rectilinear figure. The main idea is to just start in one corner and connect vertices to form triangles. Once you form a triangle, you can ignore that part of the figure, and just work with what is left. When you have formed all the

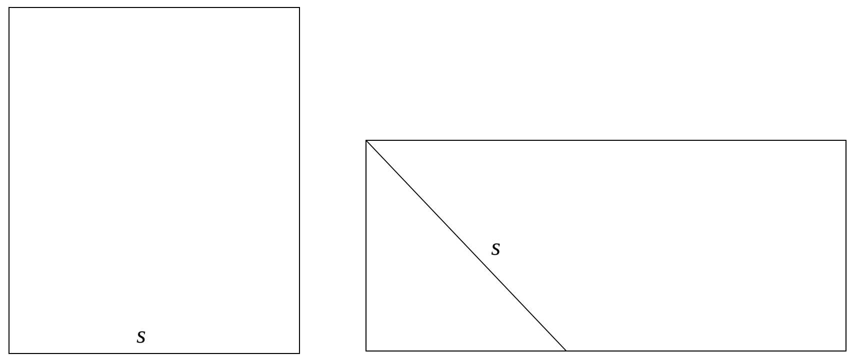
possible triangles from one vertex, move to the next vertex. Continue forming triangles and moving around the figure (move consistently clockwise or counterclockwise).

Problem 5 *(Student page 83)* This is a straightforward application of the algorithm from the previous investigation. The pictures below show the steps involved:



EQUIDECOMPOSABLE FIGURES *(continued)*

Problem 6 *(Student page 83)* Again, this is just an application of the algorithm from the previous investigation. Let s be the base of the left-hand rectangle. The easiest way to proceed is to turn the right-hand rectangle on its longer side, and dissect it into a rectangle with base s.



Problem 7 *(Student page 84)* Essentially, you have two algorithms: one to dissect figure A into a rectangle, and one to dissect figure B into the same rectangle. To dissect figure A into figure B, you first go from A to the rectangle, and then see if you can "undo" the second algorithm to go back from the rectangle to figure B.

Problem 8 *(Student page 84)* A good way to be sure you start with the same area is to use graph paper and make sure that each figure contains exactly the same number of squares. As long as you do that, and you don't make the figures too complicated so that you get too many triangles to keep track of, you can follow the algorithm.

Problem 9 *(Student page 84)* Step 1 follows from Lemma 2. Steps 2 and 3 follow from Lemma 1, and Steps 4 and 5 follow from Lemma 3. Steps 6 and 7 are basically common sense; if two or more rectangles have the same base, they can be stacked on top of each other, forming a taller rectangle with the same base.

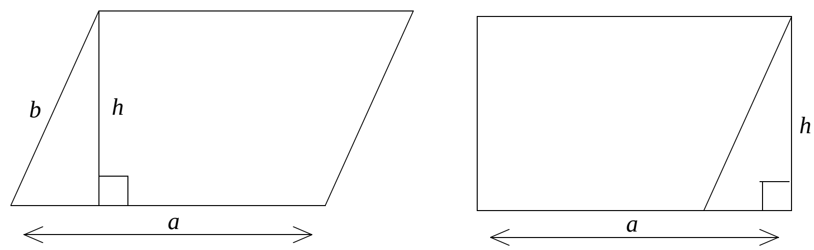
Problem 10 *(Student page 84)* The first part of the conclusion states that, because A and B had the same area, the rectangles made from them will have the same area. This is because dissection preserves area. The rectangle resulting from Step 6 will have the same area as A, while the rectangle from Step 7 will have the same area as B. Therefore, these two final rectangles have the same area as each other.

The second part of the conclusion states that because these final two rectangles have the same base and the same area, they must be congruent. Two rectangles with the same base and area must have the same height. Thus, all sidelengths are the same, so the rectangles are congruent.

The third part of the conclusion says that, because A and B have been dissected into the same rectangle, they are equidecomposable. This is due to Lemma 4.

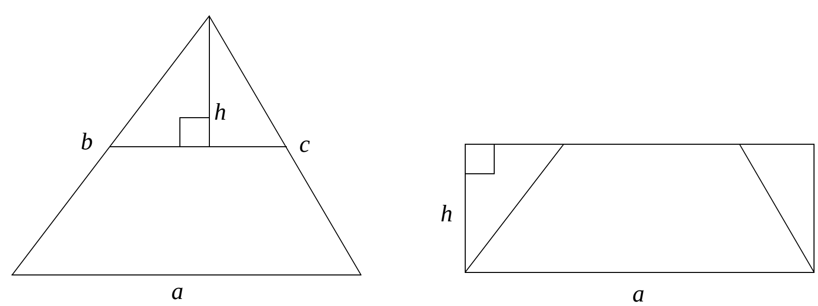
AREA AND PERIMETER

Problem 1 *(Student page 87)* Suppose you start with a parallelogram with sides of lengths a and b and height h, and dissect it into a rectangle.



The perimeter of the parallelogram is $2(a+b)$; by studying the dissection algorithm, you see that the resulting rectangle has perimeter $2(a+h)$. Notice that the height forms a right triangle in the corner of the parallelogram, and this right triangle has a leg of length h and a hypotenuse of length b. Now, the hypotenuse is always the longest side of a right triangle, so $b > h$, which implies that $2(a+b) > 2(a+h)$. So, the parallelogram has the greater perimeter.

Problem 2 *(Student page 88)* Suppose you have a triangle with sides of lengths a, b, and c, and let h be the length of the perpendicular from one of the vertices to the midline connecting the two sides that intersect at that vertex. Dissect the triangle into a rectangle.

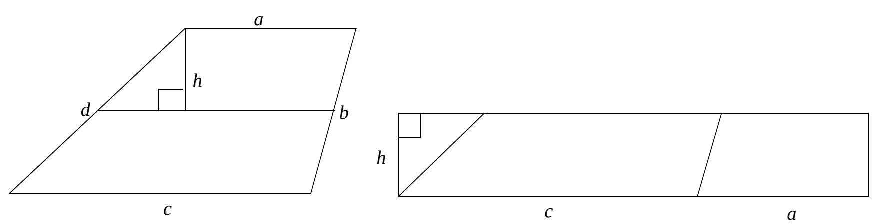


For the triangle, equal perimeters would imply that $a + b + c = 2a + 2h$, or $b + c = a + 2h$. From the cutting algorithm, all we know is that $b > 2h$ and $c > 2h$. It is possible the value of a could "make up the difference" to make the two sidelengths equal (or unequal in either direction.)

For the trapezoid, we know that $d > 2h$ and $b > 2h$. If the two perimeters are equal, then $a + b + c + d = 2h + 2a + 2c$, so $b + d = 2h + a + c$. Again, this is possible depending on the values of a and c.

The triangle has perimeter $a + b + c$, and the rectangle has perimeter $2(a + h)$. You can find examples where the rectangle has greater perimeter, yet you can also find examples where the triangle has the greater perimeter. There are even situations where the two perimeters are equal.

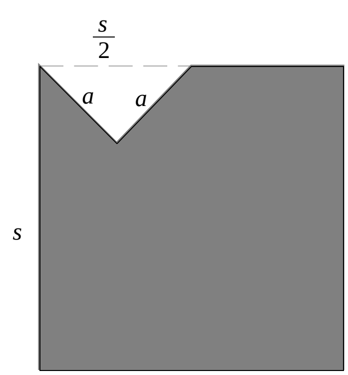
Similarly, when you dissect a trapezoid into a rectangle, the perimeter may increase, decrease, or remain constant. If the sides of the trapezoid have lengths of a, b, c, and d, and if h is the length of the perpendicular from a vertex to a midline, then the perimeter of the trapezoid is $a + b + c + d$, while the perimeter of the rectangle is $2h + 2(a + c)$.



Problem 3 *(Student page 88)* The algorithm to dissect a parallelogram into a rectangle consistently decreases the perimeter, while the algorithms to dissect triangles and trapezoids into rectangles may either increase the perimeter, decrease it, or leave it fixed (depending on the dimensions of the original figure).

The Triangle Inequality tells us that the new shape has greater perimeter than the original square. Looking at the triangles, we see that $2a > \frac{s}{2}$, and each side s is replaced by $4a$. It remains only to calculate how much it grows.

Problems 4–6 *(Student pages 89–91)* Let s be the sidelength of the original square, and let a be the length of each leg of the isosceles right triangles that you construct.



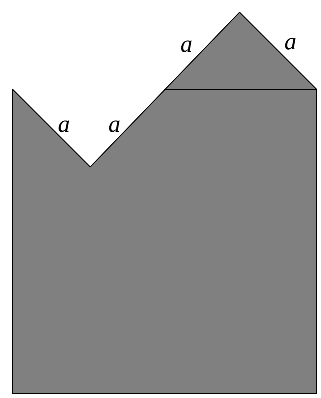
It will be helpful to know the value of a. By the Pythagorean Theorem,

$$\begin{aligned} 2a^2 &= \left(\frac{s}{2}\right)^2 \\ 2a^2 &= \frac{s^2}{4} \\ a^2 &= \frac{s^2}{8}. \end{aligned}$$

Some people make it a habit to “rationalize” the denominators in a calculation like this. It is not a necessary step, but in this case it makes a later calculation clearer.

Therefore, $a = \frac{s}{\sqrt{8}} = \frac{\sqrt{2}s}{4}$.

The new figure will clearly have the same area as the original square. To calculate the perimeter of this new figure, study how each side of the square changes. Each side of the square is replaced by four segments of length a.



Each side of length *s* is replaced by segments that total 4*a* in length.

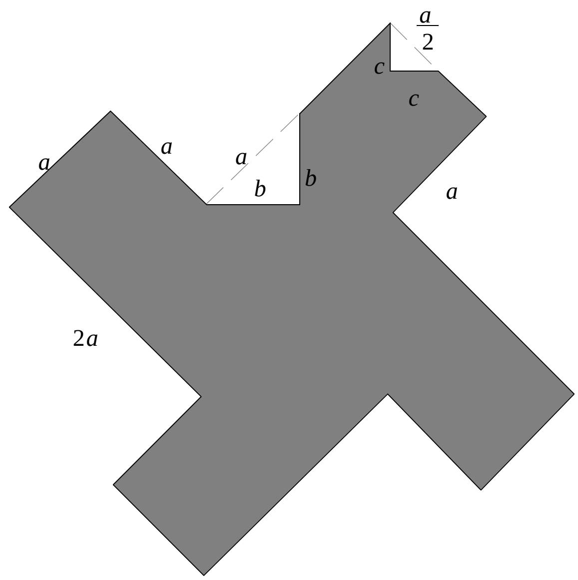
It follows then that the new perimeter is equal to $4(4a)$, since each of the four sides of the square change in the same way. This is equal to $16a = 16\frac{\sqrt{2}s}{4} = 4\sqrt{2}s$. This is $\sqrt{2}$ times the perimeter of the original square.

Problems 7–9 *(Student pages 92–93)* This time the starting figure has two different sidelengths: it has 4 long sides of length $2a$, and 8 short sides of length a (all notation as in the solution for Problem 6). As in Problem 6, the best bet is to see what happens to each type of side. Let b be the length of the legs of the isosceles right

Again, the Triangle Inequality tells us the perimeter will increase.

AREA AND PERIMETER (continued)

triangle formed along a side of length $2a$, and let c be the length of the legs of the isosceles right triangle formed along a side of length a.



Notice that $2b^2 = a^2$ and $2c^2 = (\frac{a}{2})^2$.

The Pythagorean formula tells you that $2b^2 = a^2$, so

$$b = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}} = \frac{\sqrt{2}a}{2}.$$

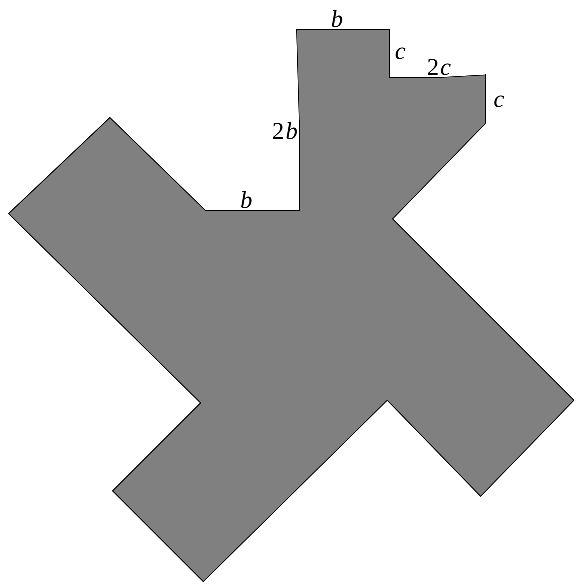
Apply the Pythagorean Theorem again to see that $2c^2 = (\frac{a}{2})^2$, so $c^2 = \frac{a^2}{8}$, and thus

$$c = \frac{a}{\sqrt{8}} = \frac{\sqrt{8}a}{8} = \frac{\sqrt{2}a}{4}.$$

Using the fact that $a = \frac{\sqrt{2}s}{4}$, you see that

$$b = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}s}{4} = \frac{s}{4} \qquad \text{and} \qquad c = \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{2}s}{4} = \frac{s}{8}.$$

Now, each side of length $2a$ is replaced by 4 segments of length b, and each side of length a is replaced by 4 segments of length c.



Since the old perimeter is $4(2a)+8a$, the new perimeter is $4(4b)+8(4c) = 16b+32c$. This is equal to $16 \cdot \frac{s}{4} + 32 \cdot \frac{s}{8} = 4s + 4s = 8s$.

The new perimeter is $\sqrt{2}$ times the previous perimeter, and it is double the perimeter of the original square.

Problems 10–11 *(Student page 93)* The problems show that by cutting and rearranging, you can create a figure with greater perimeter than the original but with the same area. In fact, since the perimeter doubles for every two applications of the cutting process, there is no upper bound on the perimeter—you can create a figure with as large a perimeter as you want and with the same area as the original square.

Starting with the square, the perimeter went from $4s$ to $4\sqrt{2}s$ to $8s$. Notice that $4\sqrt{2}s \cdot \sqrt{2} = 8$. It seems as if the perimeter will increase by $\sqrt{2}$ each time, and it will double for every two applications of the cutting process.

MAKING THE MOST OF PERIMETER

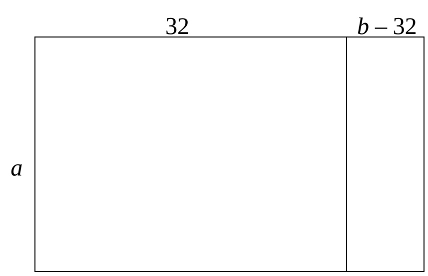
Problem 1 *(Student page 94)* If you can cut up a figure and rearrange its pieces so that they completely cover another figure, then the two figures have the same area. It makes sense then, that if the rearranged pieces of the first figure fit *entirely inside* the second figure, that the first figure has smaller area.

Problem 2 *(Student page 94)* The best rectangle for a given perimeter is a square, where"best" indicates largest area. There are several ways to investigate this, including experimental, numerical, algebraic, geometric, and analytic techniques:

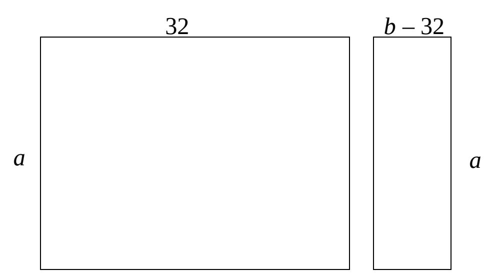
- Use some string of fixed length and a partner to look at different rectangular shapes and their respective areas, or set up a sketch using geometry software.
- Because the perimeter of any rectangle is twice the sum of the lengths of any two nonparallel sides, look at pairs of numbers whose sum is 64 and look for the pair with the greatest product. (Generalize this by thinking of the pairs of numbers as $(32+1)(32-1)$, $(32+2)(32-2)$, $(32+3)(32-3)$, and so on; or, for any sum, $(x+a)(x-a)$, $(x+b)(x-b)$, $(x+c)(x-c)$, and so on.)
- Set up an algebraic expression which gives the area of the rectangle as a function of one dimension (the perimeter). Then graph the resulting expression and locate its maximum either by estimation or by analytic techniques.
- Work through a geometric proof of the fact that the square is the best rectangle for a given perimeter. This is done in the Student Module.

Some methods yield a precise answer. Others may yield a conjecture to motivate the search for a reason, a proof, or precision.

Problem 3 *(Student page 95)* The same argument works to show that any non-square rectangle with perimeter 128 has smaller area than the 32×32 square. Suppose the rectangle has dimensions $a \times b$, where $b > a$. Then it follows that b is greater than 32, while a is less than 32. Notice that $2(a+b) = 128$, so $a+b = 64$. The resulting pictures are:



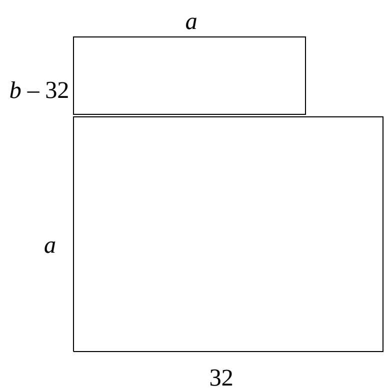
Snip!



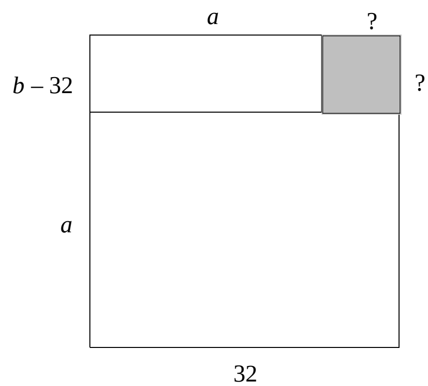
Then rearrange these . . .

Why is this a square? That would mean that $a + (b - 32) = 32$. We know $a + b = 64$, so $a = 64 - b$. Substitute to get $a + (b - 32) = 64 - b + b - 32 = 32$.

Then, take the small strip off the side, and put it on top of the $a \times 32$ rectangle:



. . . to get this . . .

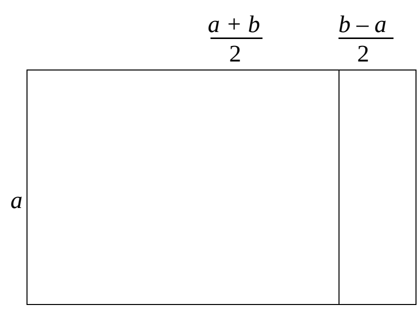


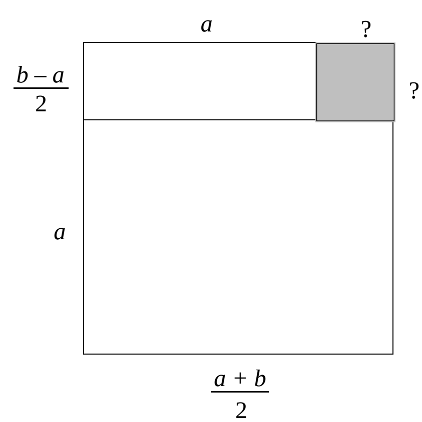
. . . which is not quite a square.

The final picture shows that the area of the $a \times b$ rectangle is less than the area of the 32×32 square.

Note that $\frac{b-a}{2} + a = \frac{a+b}{2}$.

Problem 4 *(Student page 95)* Go through the construction outlined in the previous problem, but this time do it in full generality. Suppose the rectangle has dimensions $a \times b$ with $a < b$. The side of the square in this case is $\frac{a+b}{2}$.





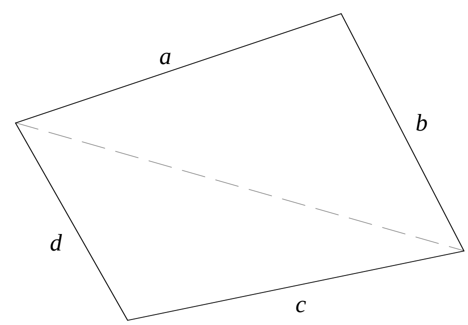
Here's another way to show that the area of any $a \times b$ rectangle is smaller than the area of the square with the same perimeter:

Since $a < b$, $b - \frac{p}{4} > 0$ and $\frac{p}{4} - a < 0$ and $c > 0$.

If p is the perimeter of the rectangle with sides of length a and b, then $p = 2(a+b)$. The sidelength of the square with perimeter p is $\frac{p}{4}$. Now, $a+b = \frac{p}{2}$ implies $b - \frac{p}{4} = \frac{p}{4} - a$. Let $c = b - \frac{p}{4}$; then also $c = \frac{p}{4} - a$. Therefore,

$$\begin{aligned}\text{area of rectangle} &= ab \\ &= \left(\frac{p}{4} - c\right)\left(\frac{p}{4} + c\right) \\ &= \left(\frac{p}{4}\right)^2 - c^2 \\ &= \text{area of square} - c^2 \\ &< \text{area of square.}\end{aligned}$$

Problem 5 *(Student page 96)* This time you want to look at *all* quadrilaterals with perimeter 128, and show that the square still has the most area. The trick here is to think of the quadrilateral as the union of two triangles, and maximize the area of each triangle. Suppose you have a quadrilateral with sides of length a, b, c, and d.

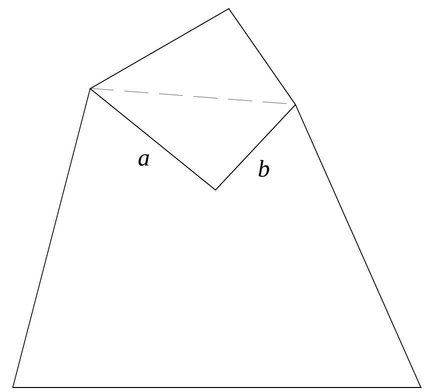


Look at the the top triangle. It turns out that of all triangles with sidelengths a and b, the one with the largest area is the one with a right angle included between a and b. You can see this experimentally, with geometry software. The point is that the area of the triangle will be half the product of one of the given sides and the height, and the height is a maximum when it coincides with the other given side.

So each triangle has the greatest area when it is a right triangle, and the quadrilateral with the greatest area will be a rectangle. But you already know that of all rectangles with fixed perimeter, the square has the largest area.

MAKING THE MOST OF PERIMETER (continued)

Problem 6 *(Student page 96)* Any polygon with a concavity can be changed to one with greater area and same perimeter by removing the concavity, as shown below:

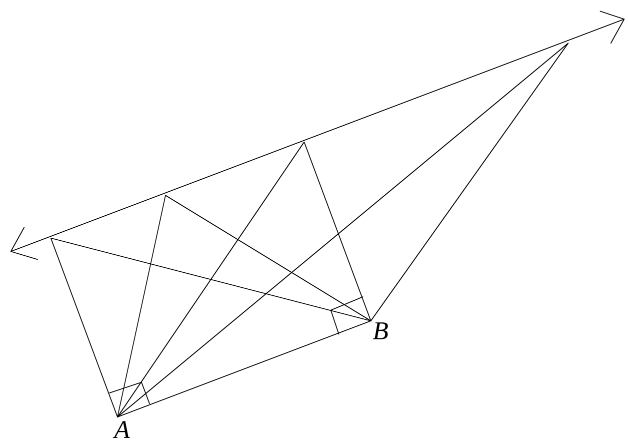


Reflect sides a and b about the dashed line.

Problem 7 *(Student page 96)* An isosceles triangle will give the smallest perimeter for fixed base and height. This is a natural guess for a couple of reasons:

- The square is a "regular quadrilateral," and it minimizes perimeter for fixed area. Perhaps regular polygons minimize perimeter for fixed area and fixed number of sides. (In fact, this is true.) In this case, the equilateral triangle would be best, but since you can't necessarily make an equilateral triangle with the given base and height, the isosceles triangle is as close as you can get.
- Think about a geometry experiment with triangle base $\overline{AB}$, a line parallel to $\overline{AB}$ a fixed distance (the height of the triangle) away, and a moving point on that line. If the point moves past the two endpoints of $\overline{AB}$, the perimeter of the triangles

grows without bound. So the minimal perimeter happens somewhere between the two right triangles; the "middle" seems like a natural place to check.



Problem 8 *(Student page 96)* This is a difficult problem that is taken up in the module *Optimization*. The theorem, known as the Isoperimetric Theorem, states that for a given perimeter, the circle is the curve that encloses the most area. An outline for the proof is as follows:

- For a fixed perimeter, show that there is at least one curve that encloses the most area.
- Given a curve that encloses the most area, show that any line that cuts its perimeter in half also bisects its area. Call such a line of symmetry a "diameter" of the curve.
- Given a curve that encloses the most area and given one of its diameters, show that, if you pick any point on the curve and draw segments from it to the endpoints of the diameter, the angle you get is a right angle.

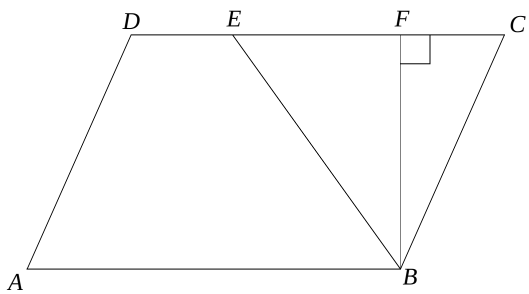
Another possible approach is to follow this logic:

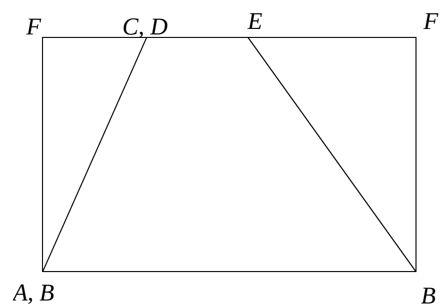
- For a given number of sides n and for a fixed perimeter, show that the regular n-gon encloses the most area.
- For a given perimeter, show that the regular $(n+1)$-gon of that perimeter encloses more area than the regular n-gon.
- Consider the circle to be the limit of a sequence of regular n-gons. It follows (with some hand waving) that the circle encloses more area for a fixed perimeter than any regular polygons.

This doesn't show (while the above outline does) that the circle is better than, say, an ellipse.

Analyzing Dissections

Problem 1 *(Student page 98)* The triangle did not have to be isosceles. To see this, first label the vertices as shown below, and label the corresponding vertices in the resulting rectangle.

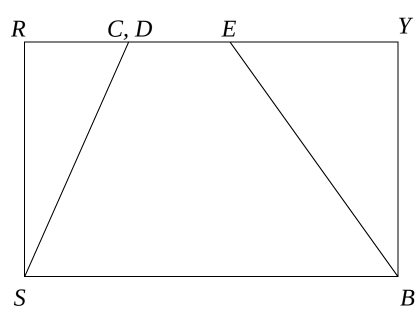




In order for the pieces to fit together properly, you need $AD = CB$, which follows automatically, since $ABCD$ is a parallelogram. In fact, notice that when $\triangle ECB$ is cut into two triangles, $\triangle EFB$ is put right back where it originally came from. So $\triangle FCB$ is the only piece that is really moved.

Problem 2 *(Student page 98)* An altitude in a triangle is a perpendicular segment from one of the vertices to the side opposite that vertex. Ann needed to cut along an altitude in order to obtain two right angles to use in forming her rectangle. Without a protractor, Ann could have constructed an altitude by folding $\triangle ECB$ at vertex B, lining up $\overline{CF}$ with the side containing $\overline{EF}$. Since EFC is straight, the angle at F is 180°. That angle is bisected by the fold, so the two angles at F ($\angle CFB$ and $\angle EFB$) will each be 90°.

Problem 3 *(Student page 98)* You want to show that angles $\angle RSB$ and $\angle YBS$ are right angles.



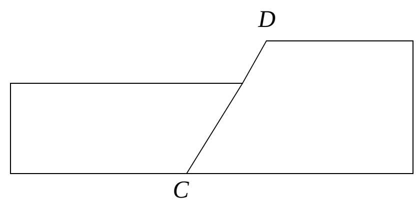
Consider $\angle RSB$. Looking at the earlier figures from Problem 1, you see that this angle comes from two other angles: $m\angle RSB = m\angle FBC + m\angle DAB$. Because opposite angles in a parallelogram are congruent, $m\angle DAB = m\angle FCB$. Therefore, $m\angle RSB = m\angle FBC + m\angle FCB$. This sum is equal to 90°, since $\angle FBC$ and $\angle FCB$ are the two acute angles in right triangle $\triangle FCB$. Thus, $\angle RSB$ is a right angle. This implies that $\angle TBS$ is also a right angle, since angles R and Y in $RSBY$ are right angles, and the sum of the measures of the angles in a quadrilateral is 360°.

Problem 4 *(Student page 100)* If segment $\overline{AB}$ makes a right angle with the base of the parallelogram at A, then it must also make a right angle with the top of the parallelogram at B. This is because opposite sides of a parallelogram are parallel, so a perpendicular to one of them is also perpendicular to the other.

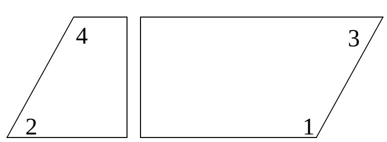
Problem 5 *(Student page 100)* Any *quadrilateral* with four right angles is a rectangle, so Peter needs to establish that his new figure has four sides, as well as four right angles.

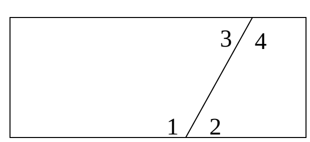
Problem 6 *(Student page 100)* The two edges joined at $\overline{CD}$ were opposite sides of a parallelogram, and thus have the same length.

Problem 7 *(Student page 100)* If the joined edges had different lengths, then Peter's new figure would not have four sides. It would look something like this:



Problem 8 *(Student page 100)* The figures below show the numbered angles in the rectangle and the corresponding angles in the original parallelogram so that you can see where these angles came from.



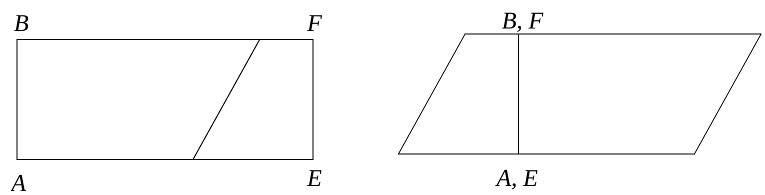


Look at the original figure. In a parallelogram, opposite angles are congruent and consecutive angles are supplementary, so you know that

$$m\angle 1 + m\angle 2 = 180^\circ \qquad \text{and} \qquad m\angle 3 + m\angle 4 = 180^\circ.$$

This shows that when placed alongside each other in the rectangle, both pairs of angles form straight lines.

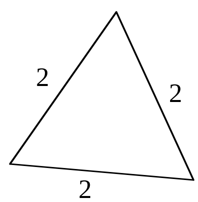
Problem 9 *(Student page 101)* If Peter's "rectangle" is labeled $ABEF$, you can trace through his algorithm and see which vertices of the original parallelogram correspond to A, B, E, and F. You will see the following:



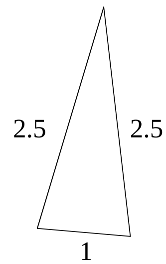
Hence, $AB = EF$ because they came from the same segment in the original parallelogram. $\overline{AE}$ in the rectangle is made up of two pieces, and these two pieces form the base of the parallelogram. Likewise, $\overline{BF}$ in the rectangle also consists of two pieces, and these two pieces form the top of the parallelogram. Since opposite sides of a parallelogram are congruent, it follows that in the rectangle, $AE = BF$.

Problem 10 *(Student page 102)* Jeremy's goal was to create $\triangle DEF$ with the same *area* as $\triangle ABC$, but with no angles of the same measure. What he actually did was create $\triangle DEF$ with the same *perimeter* as $\triangle ABC$, and with no angles congruent. Is this the same thing? In other words, does the fact that $\triangle DEF$ has the same perimeter as $\triangle ABC$ guarantee that $\triangle DEF$ and $\triangle ABC$ also have the same

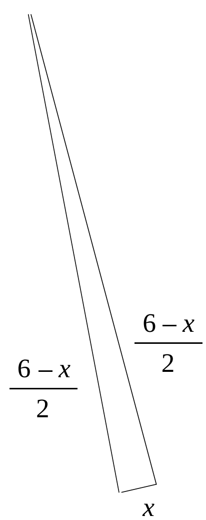
area? No! We have seen many examples of ways to change figures so that perimeter grows while area either remains fixed or even decreases. Here is an example of two triangles with the same perimeter and different areas.



perimeter = 6
area = $\sqrt{3}$
(about 1.7*)*



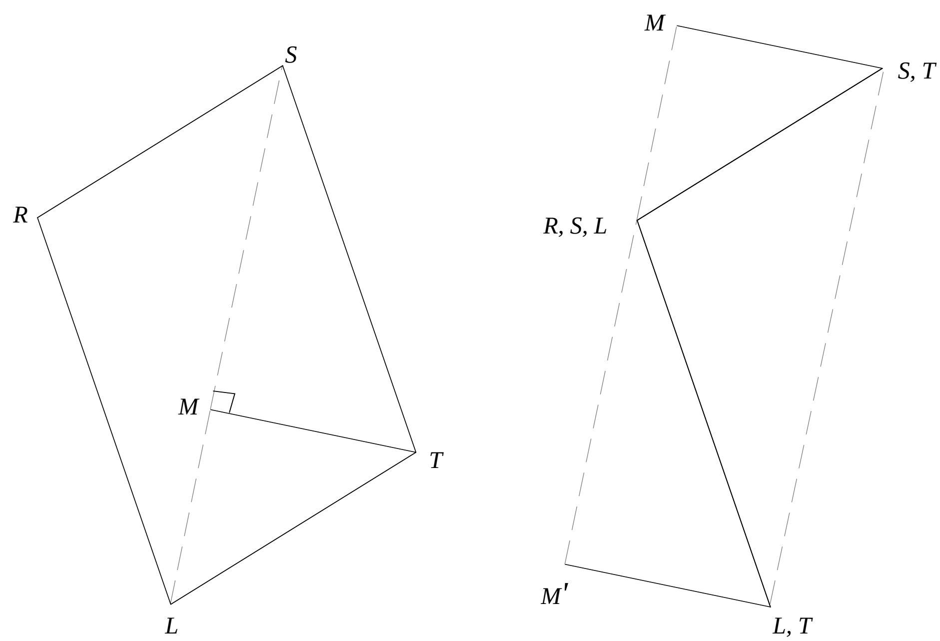
perimeter = 6
area = $\frac{\sqrt{6}}{2}$
(about 1.2*)*



You can think of starting with the equilateral triangle and “redistributing” the length of one of the sides almost entirely to the other two sides, so you are left with just a sliver of a triangle.

ANALYZING DISSECTIONS *(continued)*

Problem 11 *(Student page 102)* Start with parallelogram *RSTL*, with diagonal $\overline{LS}$. Draw a perpendicular from T to $\overline{LS}$, intersecting $\overline{LS}$ at M.



Cut out $\triangle LMT$ and move it to the side of $\triangle LRS$, lining up sides $\overline{LT}$ and $\overline{RS}$, which have the same length. Then cut out $\triangle SMT$, and move it to the other side of $\triangle LRS$, lining up congruent sides $\overline{ST}$ and $\overline{RL}$.

Is the new figure a rectangle? To show that it is a quadrilateral (rather than a pentagon), we must show that M, R, and M' are collinear, or, equivalently, that $\angle MRM'$ is a straight angle, that is, $m\angle MRM' = 180^\circ$. Notice that $\angle MRM'$ is made up of the following three angles from the figure on the left: $\angle TSL$, $\angle TLS$, and $\angle SRL$. Since $\angle SRL$ and $\angle LTS$ are opposite angles of a parallelogram, they are congruent. Thus, we have

$$\begin{aligned} m\angle MRM' &= m\angle TSL + m\angle TLS + m\angle SRL \\ &= m\angle TSL + m\angle TLS + m\angle LTS \\ &= 180^\circ, \end{aligned}$$

since $\angle TSL$, $\angle TLS$, and $\angle LTS$ are the three angles of ΔSLT. Therefore, we know that M, R, and M' are collinear. Also, there is no "overhang" because opposite sides of parallelogram $RSTL$ were matched up, and opposite sides of any parallelogram are congruent.

It is a parallelogram, since opposite sides are congruent. ($\overline{M'M} \cong \overline{LS}$ since they were both the diagonal of the original parallelogram. $\overline{M'T} \cong \overline{MT}$ since they were made by the same cut.)

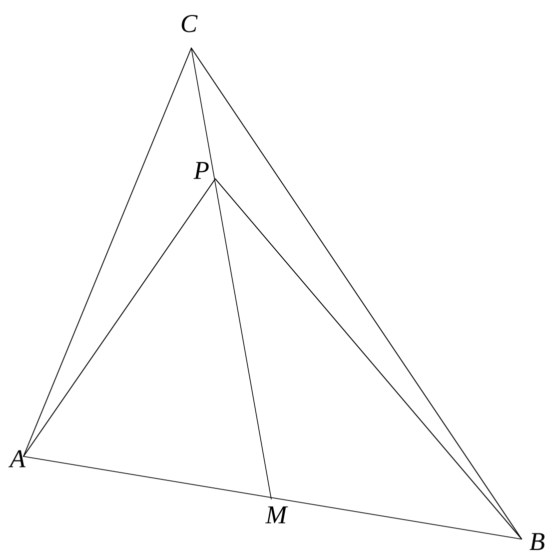
Angles M and M' are right angles, since $\overline{TM}$ and $\overline{LS}$ were perpendicular in the figure on the left.

In a parallelogram, opposite angles are congruent, so we can conclude that all four angles are right angles. Therefore, the new figure is indeed a rectangle.

There's another way to approach this problem: Let d be the length of the parallelogram's diagonal. You know you can dissect the parallelogram into a rectangle. Then, dissect that rectangle into a rectangle with base d, as you can always transform any rectangle to any other rectangle with a specified base.

To show two rectilinear figures are scissors-congruent, you just need to show that they have the same area. That is what the Bolyai-Gerwien Theorem states.

Problem 12 *(Student page 103)* From Problem 12 of Investigation 3.7, you know that any median divides a triangle into two triangles with equal area. In this case, since $\overline{CM}$ is a median of $\triangle ABC$, it follows that Area($\triangle ACM$) = Area($\triangle BCM$).



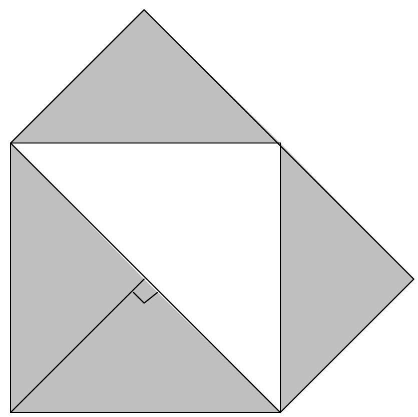
Notice that $\overline{PM}$ is a median in $\triangle APB$, so Area($\triangle APM$) = Area($\triangle BPM$). This implies that

$$\text{Area}(\triangle ACM) - \text{Area}(\triangle APM) = \text{Area}(\triangle BCM) - \text{Area}(\triangle BPM),$$

which means that Area($\triangle APC$) = Area($\triangle BPC$). Since these two figures have the same area, they are scissors-congruent.

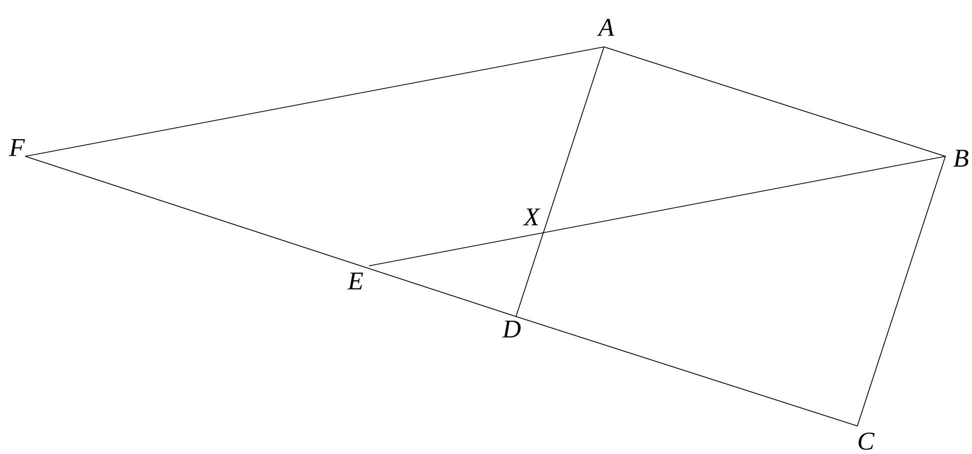
Problem 13 *(Student page 103)* You can use the usual algorithm for dissecting a triangle into a rectangle, choosing the longest side of the triangle as the base for your rectangle. Since the height to the longest side is the smallest height in the triangle, and since the rectangle you construct will have $\frac{1}{2}$ that height, you will get a rectangle with height smaller than any of the three heights of your triangle.

Problem 14 *(Student page 104)* Since a square is also a parallelogram, you can use the same technique as in Problem 11.



Alternatively, let d be the length of the square's diagonal. Remember that you have an algorithm to dissect any rectangle into another rectangle having a base of length d. But you can apply this algorithm to the square itself, since a square is a rectangle.

Problem 15 *(Student page 104)* To show that $BXDC$ is scissors-congruent to $AXEF$, it suffices to show they have the same area.



Breaking up the areas, you see that

$$\text{Area}(BXDC) = \text{Area}(ABCD) - \text{Area}(AXB)$$

$$\text{and} \quad \text{Area}(AXEF) = \text{Area}(ABEF) - \text{Area}(AXB).$$

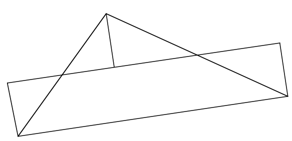
Thus, you will be done if you can show that *ABCD* and *ABEF* have the same area.

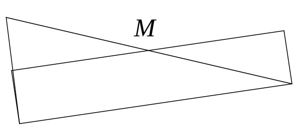
Since the areas of both rectangles and parallelograms are given by base times height,

$$\text{Area}(ABCD) = DC \cdot AD, \quad \text{and} \quad \text{Area}(ABEF) = FE \cdot AD.$$

But you can do some substituting here. Since opposite sides of rectangles and parallelograms are congruent, $FE = AB = DC$. This implies that $\text{Area}(ABEF) = DC \cdot AD$, so *ABCD* and *ABEF* have the same area. This shows that $\text{Area}(BXDC) = \text{Area}(AXEF)$.

One way to actually dissect a generic triangle into a right triangle of the same area is to go via a rectangle:

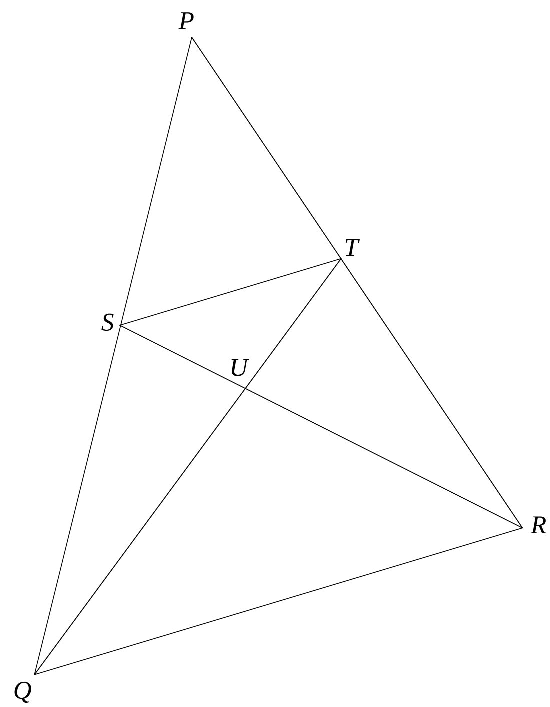




Problem 16 *(Student page 105)* Triangle *ABC* will be scissors-congruent to any triangle having the same area. So, if $\overline{BD}$ is the altitude to $\overline{AC}$, then $\triangle ABC$ has area $\frac{1}{2}AC \cdot BD$. A right triangle with legs of length *AC* and *BD* will have the same area, and so $\triangle ABC$ will be scissors-congruent to that right triangle.

Problem 17 *(Student page 105)* You can show that $\triangle SQU$ has the same area as $\triangle TRU$, and so the two are scissors-congruent. From the figure,

$$\begin{aligned} \text{Area}(\triangle SQU) &= \text{Area}(\triangle SQR) - \text{Area}(\triangle QUR) \\ \text{and} \qquad \text{Area}(\triangle TRU) &= \text{Area}(\triangle TQR) - \text{Area}(\triangle QUR). \end{aligned}$$

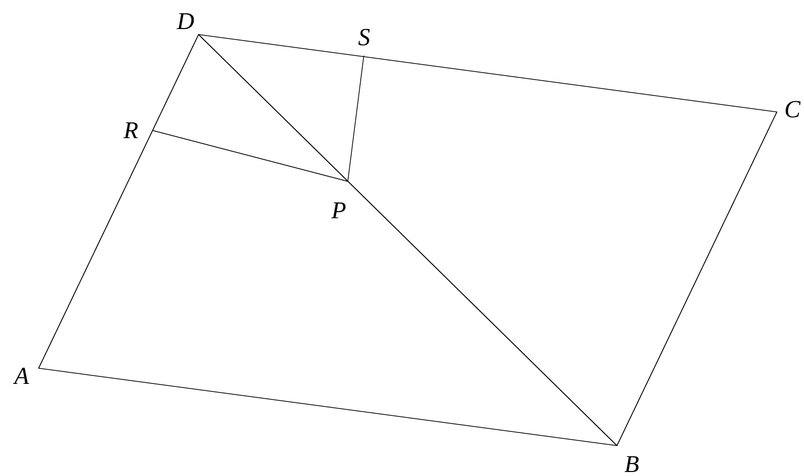


Triangles SQR and TRQ share base $\overline{QR}$. Furthermore, since $\overline{ST}$ is parallel to $\overline{QR}$, the perpendicular distance from S to $\overline{QR}$ is the same as the perpendicular distance from T to $\overline{QR}$. Thus, the two triangles have the same base and height, and thus the same area, so the equations above show that $\text{Area}(\triangle SQU) = \text{Area}(\triangle TRU)$.

Problem 18 *(Student page 106)* A diagonal of a parallelogram divides it into two congruent triangles. This means that $\text{Area}(\triangle DAB) = \text{Area}(\triangle DCB)$. Now consider $\triangle DPR$ and $\triangle DPS$. They share base $\overline{DP}$, and you are given that the perpendicular distances from S to $\overline{DP}$ and from R to $\overline{DP}$ are equal. This implies that $\text{Area}(\triangle DPR) = \text{Area}(\triangle DPS)$. It follows that

$$\text{Area}(\triangle DAB) - \text{Area}(\triangle DPR) = \text{Area}(\triangle DCB) - \text{Area}(\triangle DPS).$$

In other words, Area$(BARP)$ = Area$(BCSP)$, implying that the two figures are scissors-congruent.



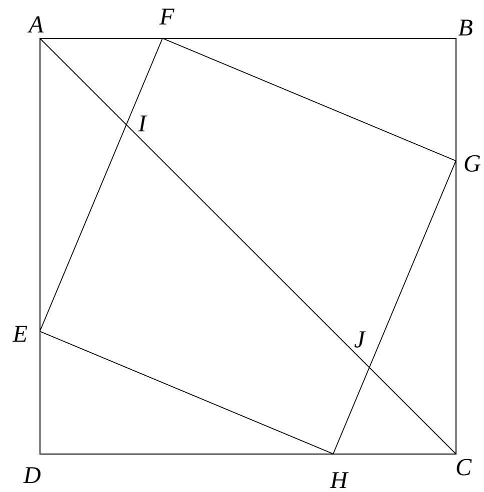
Problem 19 *(Student page 106)* Rectangles $ABCD$ and $AEFG$ must have the same area, since they are scissors-congruent. Break up each of these rectangles into two pieces:

$$\text{Area}(AGXB) + \text{Area}(GDCX) = \text{Area}(AGXB) + \text{Area}(XBEF).$$

This shows that Area$(GDCX)$ = Area$(XBEF)$.

Problem 20 *(Student page 107)* First cut along the midline parallel to $\overline{AB}$. Rotate the top triangle 180 degrees about one of the midpoints, turning the triangle into a parallelogram. You can then turn the parallelogram into a rectangle with just one more cut: cut along an internal altitude, forming a right triangle. Slide this triangle along the side of the parallelogram that was perpendicular to the altitude.

Problem 21 *(Student page 107)* The figure consists of a large square, which is made up of a smaller square and four congruent right triangles. Label the vertices as shown below.



How do you know that *EFGH* is a square?

Notice that $\overline{AC}$ is the angle bisector of two right angles. It is clear that $\triangle ADC$ and $\triangle ABC$ have the same area. By subtracting pieces of equal area, you will be able to conclude that *IEHJ* and *IFGJ* have the same area, which will show that the angle bisector divides the interior square *EFGH* into two quadrilaterals of equal area.

Since the four right triangles are all congruent, Area($\triangle DEH$) = Area($\triangle BGF$).

Since $\triangle AEF \cong \triangle CGH$, and since $\overline{AC}$ bisects the right angles of both of these triangles, we can conclude that $\triangle AIE \cong \triangle CJG$ and $\triangle AIF \cong \triangle CJH$. Therefore,

$$\text{Area}(\triangle AIE) = \text{Area}(\triangle CJG)$$

and

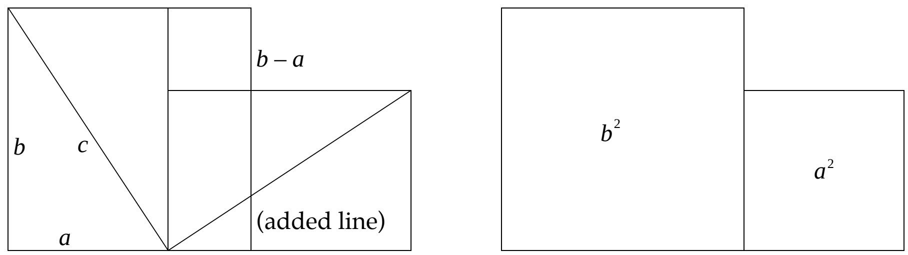
$$\text{Area}(\triangle AIF) = \text{Area}(\triangle CJH).$$

Removing all the pairs of equal-area figures from the two equal-area triangles *ADC* and *ABC*, you will be left with two figures of equal area. These two figures are precisely *IEHJ* and *IFGJ*.

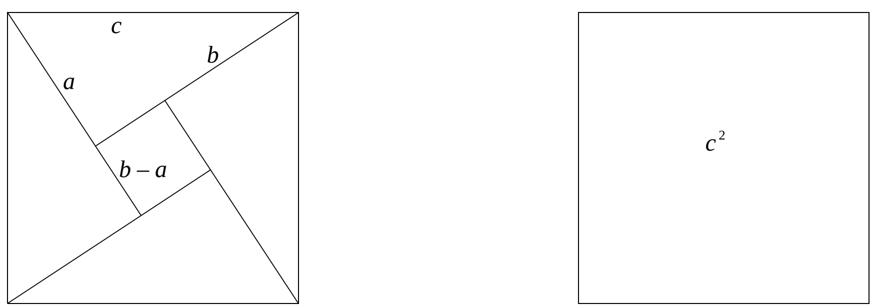
PYTHAGOREAN CUTTING PROOFS

Problem 1 *(Student page 109)* The Pythagorean Theorem says that the area of the square which lies along the hypotenuse of the right triangle is equal to the sum of the areas of the two squares which lie along the legs.

Proof 1: Adding labels to the first drawing may help, but the key insight comes in seeing where to "split" the figure into a^2 and b^2.

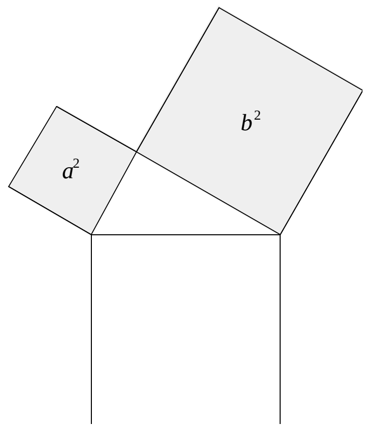


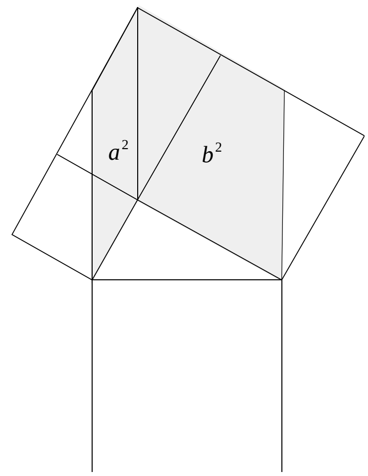
The same five pieces are used to construct c^2, so the areas of $a^2 + b^2$ and c^2 must be equal.



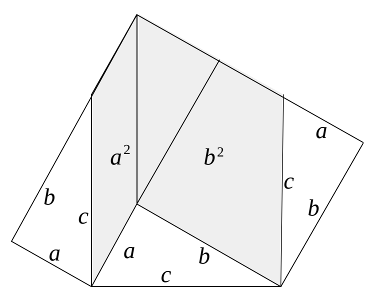
PYTHAGOREAN CUTTING PROOFS *(continued)*

Proof 2: In the first three pictures, the two parallelograms have the same areas as the two shaded squares because they have the same bases and heights.



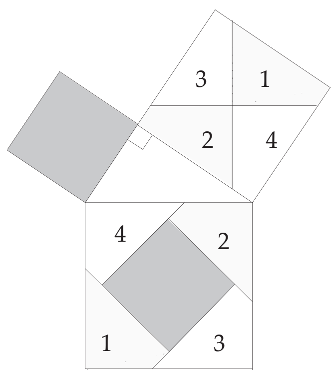


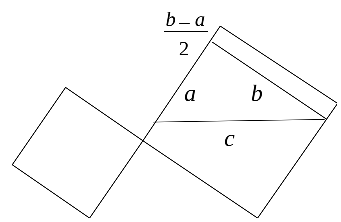
To move to the fourth picture, we need to figure out why the longer sides of the parallelograms will have length c. If you look at the unshaded triangles on the left and right, you see that they are both right triangles with legs of length a and b. They must therefore be congruent to the original triangle and have hypotenuses (the longer sides of the parallelograms) of length c.



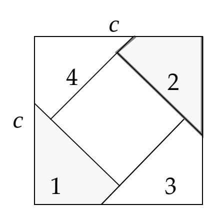
The final step works becuase of congruent triangles as well. The unshaded triangle in the fourth picture was the original triangle. Therefore, the shaded original triangle in the fourth picture is congruent to the unshaded triangle in the final picture, so the move to the fifth picture is valid.

Proof 3: Figuring out how to make the cut, finding the right lengths, and keeping track of the pieces all play a role in this proof. The first step—realizing where the cuts were made—may be the most difficult for some people. It helps to realize that none of the pieces need to be rotated for the rearrangement into the large square. Let's number the pieces. From this we can tell that the cuts through the interior of the square with sidelength b must have length c, so we can try to find our original triangle somewhere inside this square.



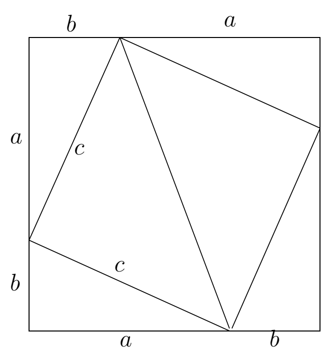


We can now see where the cuts are made: at a length of $\frac{b-a}{2}$ along the side, and through the center of the square. The final question is why all the pieces fit together. We already know that the side of length c comes from the cut through the center of the square, so we can line up the four pieces from the square with sidelength b along the outside. Why is the center shape the square with sidelength a?



The long side of the four pieces is $\frac{b-a}{2} + a$ and the shortest side is $\frac{b-a}{2}$ (look back at the previous pictures to see where these lengths come from). So the difference between them, the space left to be filled, is exactly a on each side. The four angles are right angles because they are supplementary to angles from the corners of the square with sidelength b, so the shape in the middle is exactly the square with sidelength a.

Another way to get the area formulas is to see this as half of an $a \times b$ square.



Proof 4: The original equation comes from equating the area of the whole figure with the sum of the areas of the three parts. The whole figure is a right trapezoid with bases a and b and height $(a+b)$. The area of this figure is $\frac{1}{2}(a+b)(a+b)$. The areas of the two right triangles with legs a and b are each $\frac{1}{2}(ab)$. The area of the third right triangle is $\frac{1}{2}c^2$. The rest of the proof follows by algebraic manipulations:

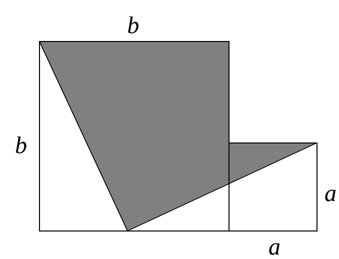
$$\begin{aligned} \frac{1}{2}(a+b)(a+b) &= \frac{1}{2}(ab) + \frac{1}{2}(ab) + \frac{1}{2}c^2 & \\ (a+b)(a+b) &= (ab) + (ab) + c^2 & \text{(multiply each side by 2)} \\ a^2 + 2ab + b^2 &= 2ab + c^2 & \text{(expand and combine terms)} \\ a^2 + b^2 &= c^2 & \text{(subtract } 2ab \text{ from each side)} \end{aligned}$$

Problem 2 *(Student page 113)* If a, b, and c are the sidelengths of the right triangle, c being the length of the hypotenuse, then the shaded portion in the drawing below has area

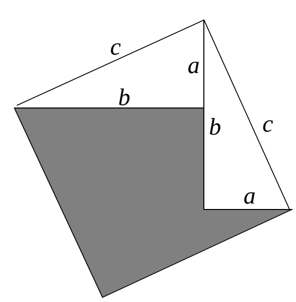
$$a^2 + b^2 - ab.$$

This is because you start with two squares, one with sidelength a, and one with sidelength b, and then you remove two copies of the right triangle. Since the right triangle has area $\frac{1}{2}ab$, you are removing a total area of ab.

The "I" in the poem is the shaded region!



Now start with this shaded region, and stand two copies of the right triangle on top of it, like this:



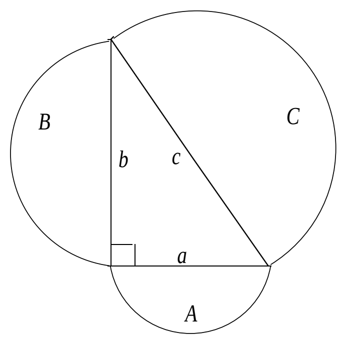
Notice that you get a square of sidelength c, so the new area is c^2. But if you calculate the area as being equal to the shaded area plus the area of two right triangles, you see it equals

$$(a^2 + b^2 - ab) + \frac{1}{2}ab + \frac{1}{2}ab.$$

Equating these two representations of the area yields

$$c^2 = (a^2 + b^2 - ab) + \frac{1}{2}ab + \frac{1}{2}ab = (a^2 + b^2 - ab) + ab = a^2 + b^2.$$

Problem 3 *(Student page 113)* Suppose you construct semicircles along the sides of a right triangle. Label the areas of the semicircles which lie on the legs A and B, and label the area of the semicircle on the hypotenuse as C. Denote by a, b, and c the sidelengths of the the right triangle.



The area of a circle is given by the formula $A = \pi r^2$, so the area of a semicircle is given by $A = \frac{1}{2}\pi r^2$. Notice that the sides of the triangle form diameters of the semicircles, so the radii are $\frac{a}{2}$, $\frac{b}{2}$, and $\frac{c}{2}$.

It turns out that the sum of the areas A and B is equal to the area C. To see this, write out these areas:

$$A + B = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{b}{2}\right)^2,$$

which is equal to

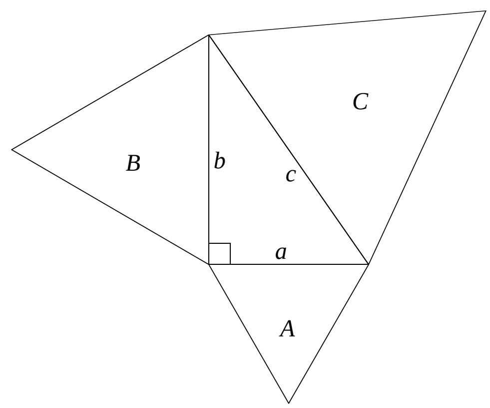
$$\frac{1}{2}\pi\left(\frac{a^2 + b^2}{4}\right).$$

But, applying the Pythagorean Theorem, this is equal to

$$\frac{1}{2}\pi\left(\frac{c^2}{4}\right) = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = C.$$

A general conclusion is that if the shapes you build on the sides of a right triangle are similar, then the sum of the areas of the shapes built on the legs will equal the area of the shape built on the hypotenuse.
The area of an equilateral triangle with sidelength s is given by $\frac{s^2\sqrt{3}}{4}$.

Now try constructing equilateral triangles along each side of the right triangle, again labeling their areas by A, B, and C.

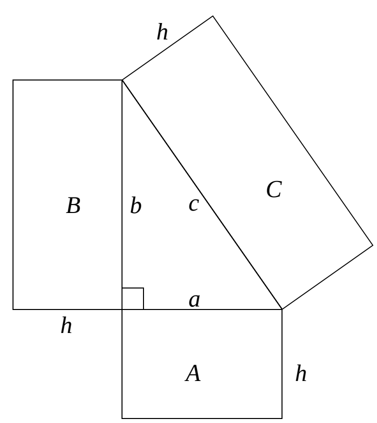


Once again it happens that $A + B = C$. You can calculate that

$$A + B = \frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} = \frac{(a^2 + b^2)\sqrt{3}}{4} = \frac{c^2\sqrt{3}}{4} = C.$$

We had to have some way to decide on the height of the rectangles. Making them all equal is one possibility. Another possibility is to make them in some appropriate ratios. That might change the answer so that $A + B \neq C$.

Now try drawing a rectangle along each side of your right triangle. Suppose each rectangle has the same height, call it h, and let A, B, and C be the three areas. This time you'll see that the sum of the areas of the rectangles on the legs of the right triangle is always greater than the area of the rectangle along the hypotenuse.



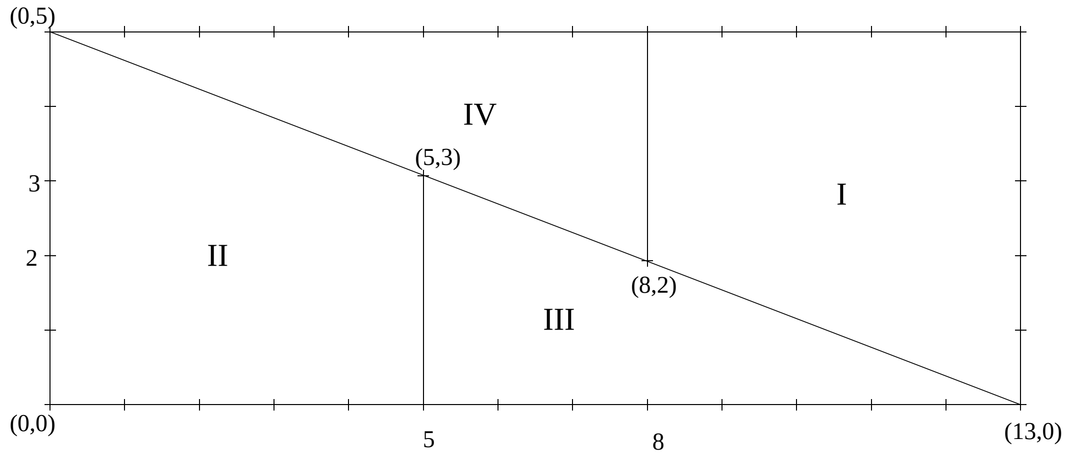
Why is this? In any triangle you have $a + b > c$. This is the Triangle Inequality. You can use this inequality to see that

$$A + B = ah + bh = h(a + b) > hc = C.$$

A CUTTING PARADOX

Problems 1–3 *(Student page 114)* It certainly looks like the four pieces A, B, C, and D exactly cover the 5×13 rectangle, but that can't be, since you know their total area is equal to 64, not 65. Look very closely at the "rectangle," and you will see that the pieces don't quite fit.

Think of the figure as lying on the xy-plane, with its lower left vertex at the origin. Since you know the sidelengths of all four interior pieces (by looking at their arrangement inside the original square), you can label the vertices of the figure which appear to lie on the diagonal of the "rectangle."



If the four pieces really form a rectangle, then the points $(0, 5)$, $(5, 3)$, $(8, 2)$, and $(13, 0)$ would all lie on the diagonal. But these four points do not all lie on the same line! The line passing through $(0, 5)$ and $(13, 0)$ has slope $-\frac{5}{13}$. The line passing between $(5, 3)$ and $(13, 0)$ has slope $-\frac{3}{8}$. The line passing between $(8, 2)$ and $(13, 0)$ has slope $-\frac{2}{5}$. So these four points cannot be collinear.

Therefore, when the four pieces are arranged as above, they do not precisely cover the 5×13 rectangle. Since the areas of the square and rectangle differ by only one unit, it is hard for your eye to detect that something is amiss. This is why it's important to be able to prove mathematically that all your dissections really work.

Problems 4–5 *(Student page 115)* The next three Fibonacci numbers are 34, 55, and 89. Each Fibonacci number is the sum of the two previous numbers. Thus, to get 34, you add 13 and 21; to get 55, you add 21 and 34; to get 89, you add 34 and 55.

A CUTTING PARADOX (continued)

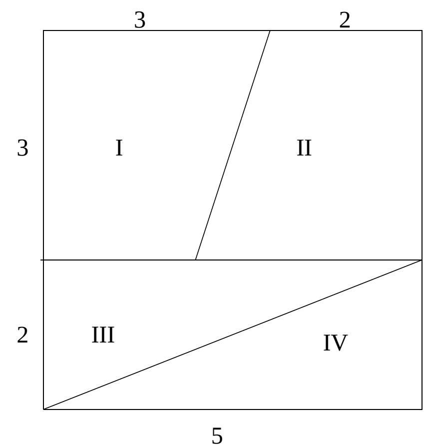
This is an example of a *recursive* definition. To find any term in the list, you need to know the terms preceeding it.

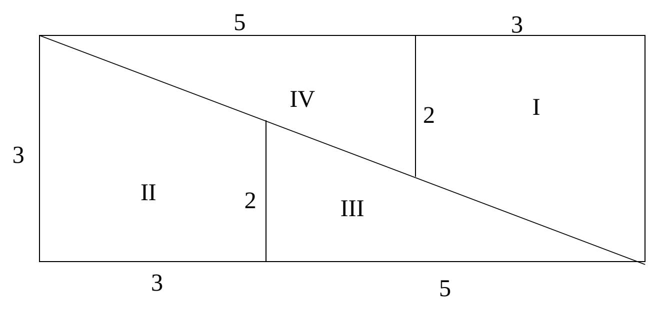
Suppose you list the Fibonacci numbers in order, as

$$f_1, f_2, f_3, \ldots, f_{n-2}, f_{n-1}, f_n, \ldots.$$

The sequence begins with $f_1 = 1$ and $f_2 = 1$. Then to find f_n for $n \geq 3$ you use the formula $f_n = f_{n-1} + f_{n-2}$.

Problems 6–7 *(Student pages 115–116)* To turn a 5×5 square into a 3×8 "rectangle," your pictures will look like this:





Suppose you pick three larger Fibonacci numbers; call them f_n, f_{n+1}, and f_{n+2}. Then you want to try and turn a square with sidelength f_{n+1} into a "rectangle" with dimensions $f_n \times f_{n+2}$. The pictures will always look similar. You want the two right triangles (pieces III and IV) to have legs of length f_{n+1} and f_{n-1} (the Fibonacci number before the first of the three you picked). The two trapezoids (pieces I and II) will have parallel sides of length f_{n-1} and f_n.

Problems 8–9 *(Student page 117)* As you fill in the chart given in the Student Module, you will notice that at the end of the nth month, the number of pairs of rabbits will be the same as the $(n+1)$st Fibonacci number. So, at the end of 12 months, there will be 233 rabbits.

Why does the Fibonacci pattern occur in this problem? Well, the first two entries are definitely 1 and 1. Now look at the number of pairs of rabbits at the end of month n for some $n > 2$. You know that every rabbit alive at the end of a month was either born that month, or was alive the previous month.

> This means that the number of pairs of rabbits at the end of month n equals the number of newborn pairs that month plus the number of pairs which were alive during month $(n-1)$.

The key fact to keep in mind is that as soon as a pair of rabbits is two months old, they have a pair of babies, and continue to do so each month. The set of rabbits having babies each month is exactly the same as the set of rabbits that have been alive for two months or longer, so the number of rabbits having babies each month is exactly the same as the number of rabbits that were alive two months ago. Substituting this into the statement above gives our final result:

> The number of pairs of rabbits at the end of month n equals the number of pairs of rabbits that were alive in month $(n-2)$ plus the number of pairs that were alive during month $(n-1)$.

If we were to write this as an equation, we would write $R_n = R_{n-1} + R_{n-2}$ where $R_1 = R_2 = 1$. But this is exactly the equation to generate Fibonacci numbers.

You can try to do a search on the Internet for Web sites dealing with Fibonacci numbers. You'll be surprised at how much you find!

Problem 10 *(Student page 117)* You can find lots of occurences of the Fibonacci numbers in nature. Many flowers have a fixed number of petals, which is a Fibonacci number. Another example is found in the seed patterns of sunflowers. The head of a sunflower is made up of a number of seeds. There is one dark seed in the middle, and the other seeds occur in spirals. The number of clockwise spirals and the number of counterclockwise spirals will both be Fibonacci numbers. Pinecones provide another example. The scales of a pinecone grow outward in a spiral pattern from the point where the cone is attached to the tree. The number of clockwise spirals and the number of counterclockwise spirals are both Fibonacci numbers.

Problem 11 *(Student page 117)* Label the entries of the new sequence by g_1, g_2, g_3, ... In other words,

$$\begin{aligned} g_1 &= 0 \\ g_2 &= 1 \\ g_3 &= 1 \\ g_4 &= 2 \\ g_5 &= 3 \\ g_6 &= 5 \\ &\vdots \end{aligned}$$

It looks like the new sequence can be described by:

$$g_1 = 0 \qquad \text{and} \qquad g_n = f_{n-1} \qquad \text{for } n > 1$$

where, as usual, f_n refers to the nth Fibonacci number.

How can you prove this? The only thing you know about the g_n's is the way they are formed: each one is formed by taking a Fibonacci number and subtracting the previous Fibonacci number. In other words, $g_n = f_{n+1} - f_n$ for all n. We know that for Fibonacci number $f_{n+1} = f_n + f_{n-1}$ for all $n > 1$. Substituting, you see that

$$g_n = (f_{n-1} + f_n) - f_n = f_{n-1}.$$

Problem 12 *(Student page 118)* The value that the ratios seem to be approaching is a famous number. It was known by the Greeks as the *golden ratio*. This number is approximately 1.618, but you can solve for it algebraically, which is what we'll do here.

Notice that each ratio in the sequence can be written as $\frac{f_n}{f_{n-1}}$. You can expand this as

$$\begin{aligned} \frac{f_n}{f_{n-1}} &= \frac{f_{n-1} + f_{n-2}}{f_{n-1}} \\ &= 1 + \frac{f_{n-2}}{f_{n-1}} \\ &= 1 + \frac{1}{\frac{f_{n-1}}{f_{n-2}}}. \end{aligned}$$

A CUTTING PARADOX (continued)

If n is very large, $\frac{f_n}{f_{n-1}} \approx \frac{f_{n-1}}{f_{n-2}}$. If we assume that these ratios really do approach a fixed value, we can call this number x, and see if we can solve for it. Let $\frac{f_n}{f_{n-1}} = \frac{f_{n-1}}{f_{n-2}} = x$ in the equation above. Then we have

$$x = 1 + \frac{1}{x}.$$

Solve for x: $x^2 = x + 1$, so $x^2 - x - 1 = 0$. Using the quadratic formula, you can calculate that $x = \frac{1 \pm \sqrt{5}}{2}$. Since all the Fibonacci numbers are positive, x must be positive, so you can conclude that the golden radio is equal to $\frac{1+\sqrt{5}}{2}$, which is approximately 1.618.

Solutions

Investigation 3.14

Student Pages 120–123

CUTTING UP SOLIDS

The only variation you can get is in the size of the circle. It can be as small as point size or as large as the circumference of the sphere.

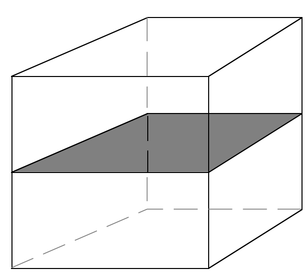
Problem 1 *(Student page 120)* No matter how you slice a sphere, you will get a circle as the cross section. To see this, first convince yourself that if you make a horizontal cut, you will get a circle. Then notice a very important property of a sphere: no matter how you rotate it, it looks exactly the same. So, if you slice at some different angle, just rotate the sphere so that the slice becomes horizontal. The cross section is then clearly a circle since the rotation didn't change the shape of the cross section, it must have always been a circle.

Problem 2 *(Student page 120)* To get a circle, slice parallel to the cylinder's circular top and base. To get a rectangle, slice perpendicular to the base. By slicing at different angles, you can get oval-shaped slices (and "cut-off ovals" if you slice through a base).

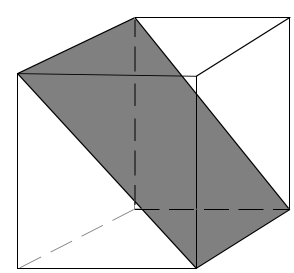
Problem 3 *(Student page 120)* The slices you can get from a cube include squares, rectangles, parallelograms, trapezoids, triangles, pentagons, and hexagons.

a. If you cut a cube parallel to any face, you will get a square.

b. To get a nonsquare rectangle, slice through any top edge and its opposite edge on the bottom of the cube.



Square

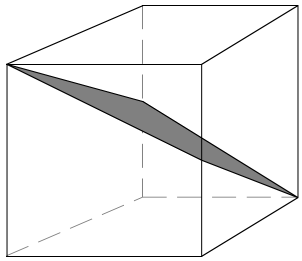


Rectangle

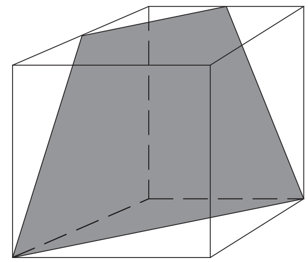
c. To obtain a nonrectangular parallelogram, choose two opposite vertices and two midpoints, choosing the midpoints so that the sides on which they lie do not have either of the chosen vertices as endpoints. Slice through these four points.

CUTTING UP SOLIDS (continued)

d. For a trapezoid, choose the midpoints of two adjacent edges of the top face of the cube and two opposite vertices on the bottom face of the cube. Slice through these four points.

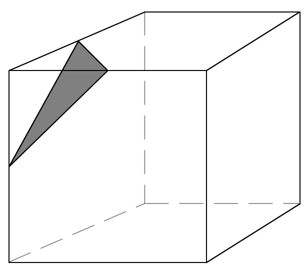


Parallelogram



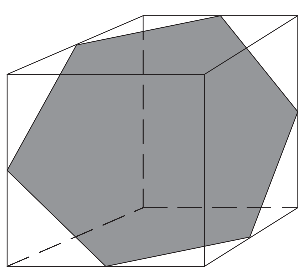
Trapezoid

e.–f. To get triangles, cut off a corner of the cube. Pick any vertex, then pick a point on each of the three edges which share that vertex as an endpoint. Slice through these three points to obtain a triangle. If you choose the midpoints of those three edges, you will get an equilateral triangle.

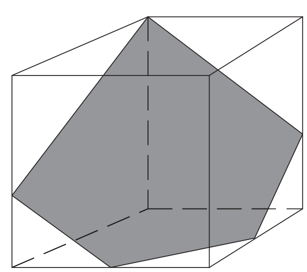


Triangle

g.–h. To get a hexagon, join six midpoints as shown below. Now picture tilting the hexagon backward, so that its two top vertices come together at one vertex. Here are ways to visualize a hexagon and a pentagon.



Hexagon



Pentagon

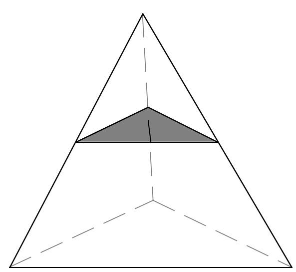
i. An octagon cannot be made because the cube has six faces, so a plane can intersect it in at most six segments. (A single plane can't intersect a face more than once.)

j. A circle is not possible because a plane must intersect faces of the cube in segments, if at all. Any cross section will consist of segments, and will thus be a polygon.

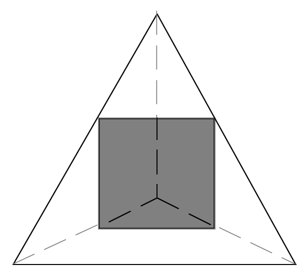
k. The only other possibilities stretch the meaning of "cross section"; the plane cannot intersect the cube at all, it can be tangent at a point, or it can be tangent at an edge.

Problem 4 *(Student page 121)* You will never be able to obtain a right angle or an obtuse angle in a triangular cross section of a cube, so right triangles and obtuse triangles cannot be made in this way. Only acute triangles are possible.

Problem 5 *(Student page 121)* Suppose you have a regular tetrahedron. If you cut parallel to any face, you will get a triangular slice. If you cut through four midpoints, as shown in the second picture, you obtain a square.



Triangle



Square

Problem 6 *(Student page 121)* The four conic sections are the circle, parabola, ellipse, and hyperbola. To get a circle, slice parallel to the circular base of the cone. A hyperbola is obtained when you slice vertically, perpendicular to the base. If you slice parallel to the side of the cone, you obtain a parabola; a slice at any other angle will produce an ellipse.

The mathematical term for a doughnut is a *torus*.

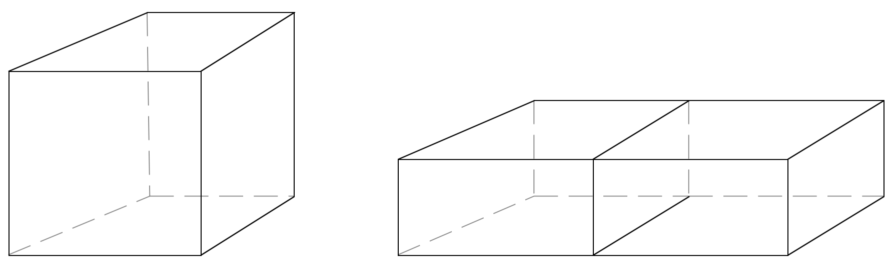
Problem 7 *(Student page 121)* If you slice the doughnut perpendicular to the table, and avoid the hole, you will get an oval-shaped cross section. If you slice parallel to the table through the entire doughnut (cutting it in half), the cross section

Volume is the three-dimensional analog of area, while surface area is the three-dimensional analog of perimeter. Remember that dissection in two dimensions did not necessarily preserve perimeter.

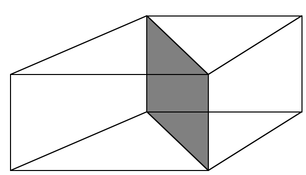
will look like an oval, with a smaller oval contained inside. If you cut through the hole, perpendicular to the table, you will get two adjacent circles.

For Discussion *(Student page 122)* When you dissect a solid figure, you should imagine that the dissection will preserve volume. Surface area need not be preserved, however. For instance, look at the figures for Problem 8 below. A cube is dissected into a box, and although the volume remains the same, the box has a larger surface area.

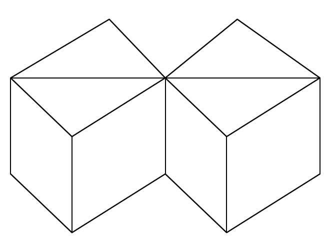
Problems 8–9 *(Student page 122)* You can slice a cube in half, slicing parallel to one of the faces; then stack the resulting two pieces on top of each other to obtain a new rectangular solid which is not a cube. You could also repeat this process any number of times, producing longer and shorter boxes.



The second shape pictured above has a right half and a left half. Take each half separately, and slice into two pieces along the shaded figure shown below:

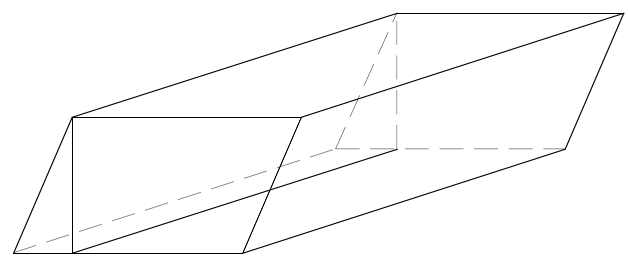


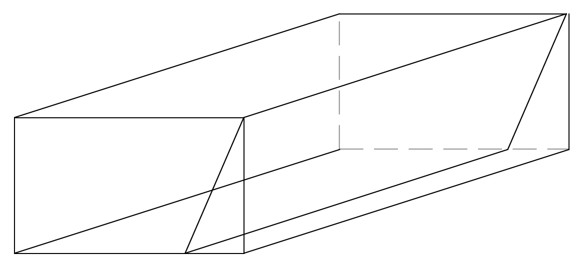
This turns each half into two wedge-shaped pieces, producing four such pieces in all. How many different ways can you join these pieces together? Here's one way:



CUTTING UP SOLIDS (continued)

Problems 10–11 *(Student page 122)* Because you can dissect a parallelogram or a triangle into a rectangle in two dimensions, it follows that you can transform a prism with a parallelogram or triangle as a base into a rectangular solid in three dimensions. Perform the same cuts as you would if you were just working with the base, but cut all the way through the prism. The pictures below show the process for a prism which has a parallelogram for a base.





Problem 12 *(Student page 122)* This is impossible; the "Perspective on the Hilbert Problems" explains why.

Problem 13 *(Student page 123)* Hilbert challenged mathematicians to find two tetrahedra of equal bases and equal altitudes which cannot be split up into congruent tetrahedra. Two such tetrahedra would have the same volume, and would not be equidecomposable. (If the first could be dissected into the second, then that means that they can be split up into congruent tetrahedra—the second wouldn't have to be split up at all.) This example would show that two solid figures with the same volume are not always equidecomposable.

Cut a Rug and Other Dissections

Problems 1–4 *(Student page 124)* There are many ways to approach these problems; the point is to realize that sectors of circles will "fold" into cone-shaped hats, but that you have to vary the size of the sector depending on how tall you want the hat to be and the size of your head. We include here some samples of what students have done.

Other groups of students gave patterns to produce hats for varying head sizes without any choice about the height of the hats. The mathematics involved in the solutions where you can choose circumference but not height is similar. Start with a large enough circle, then cut away the appropriate amount of the circle and fold what's left into a cone.

Some students like to create formulas. Here is one algebraic method for creating a witch's hat for any head size and any height:

Step 1: Measure the maximum circumference of your head. Call this number x.

Step 2: Determine how tall you want the hat to be. Call this number y.

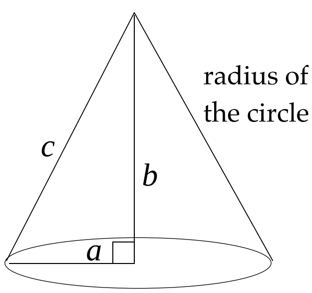
Step 3: Now for some calculations. Substitute x into this formula: $\frac{x}{\pi} = D$. When you find D, divide it by 2. Call this number z.

Step 4: Substitute y and z into this formula: $y^2 + z^2 = p^2$. Find p.

Step 5: Next you need to make a circle that has radius p.

Step 6: Now you need to cut away a piece of the circle and roll it into a cone. You want to leave enough circumference to go around your head, so you need a piece with central angle measuring $\frac{x}{2\pi p}(360^\circ)$.

Step 7: Roll it so it makes a cone, and tape the sides together. Put it on!!



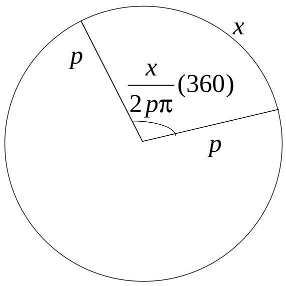
This doesn't tell you, however, where these mysterious formulas come from. Another group's work might help clarify things a bit:

You need to know the radius of the base and the height of the cone in order to use the Pythagorean Theorem to figure out the radius of the circle you will cut out.

- If you want the base to be 20 inches around, you have to figure out the radius of the base using the formula for the circumference of a circle, $C = 2\pi r$. If $2\pi r = 20$, $r = \frac{10}{\pi}$, so the radius is $\frac{10}{\pi}$, or about 3.2 inches.
- If you want a cone with a height of 10 inches, then you need to find the hypotenuse of a right triangle with legs of lengths 10 inches and 3.2 inches.

$$10^2 + (3.2)^2 = 100 + 10.24 = 110.24,$$

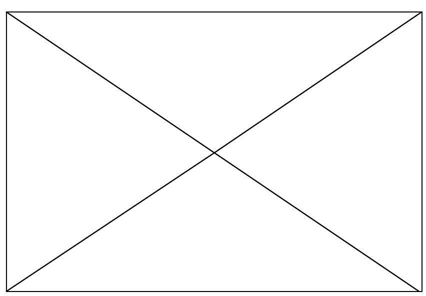
so take the square root of 110.24 to get about 10.5 inches. This is the radius of the circle you will use.

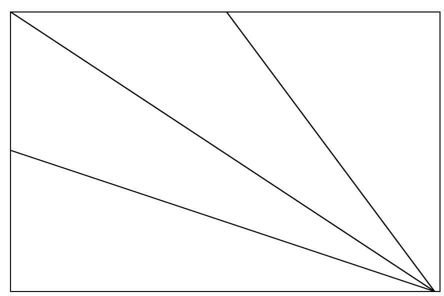


This second group didn't explain how to find the piece to cut away. So where does the central angle of $\frac{x}{2\pi p}(360^\circ)$ in Step 6 come from? The arc length you want is x. The circumference of the circle is $2\pi p$. The total central angle of a circle is 360°. So you take the fraction of the area you want to leave and multiply that by 360 to find the angle you need. This is an arc length calculation in reverse; you know the arc length (x) and want to find the central angle.

Problem 5 *(Student page 124)* You can just think of the cake as a rectangle. If you can divide the rectangle into a particular number of same-size pieces, then you can also divide the cake. To cut the cake into four same-size pieces, you can always make three cuts, dividing it into four equal horizontal or vertical strips. Or, you can divide the rectangle into four congruent rectangles some other way, such as connecting the midpoints of opposite sides.

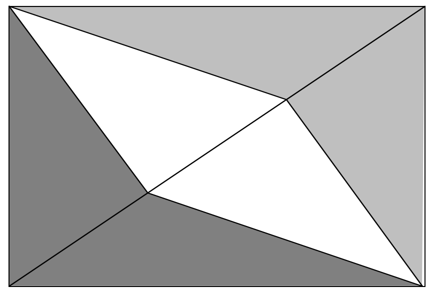
Another method is to first cut along a diagonal. This divides the rectangle into two congruent triangles; you can then divide each triangle into pieces having equal area (such as by drawing a median, as in Problem 12 in Investigation 3.7).



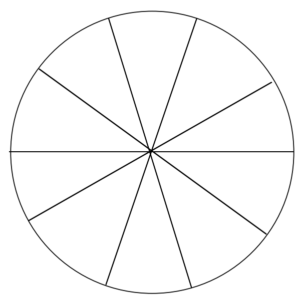


To cut the cake into three same-size pieces, you can cut it into three congruent horizontal or vertical strips. Here's a very strange way to get three equal-sized pieces;

the shading indicates the different pieces. Can you explain why they are all the same size? (Hint: First draw the diagonal of the rectangle.)

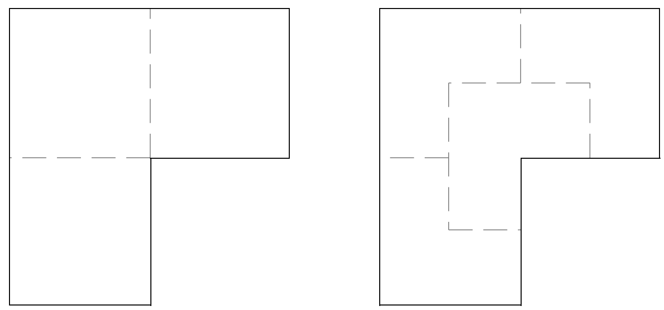


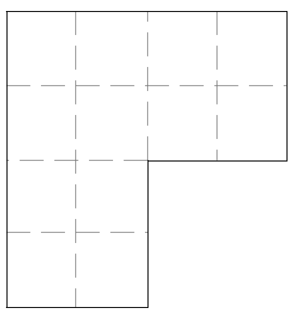
Problem 6 *(Student page 125)* The circular cake can be cut into 10 same-size pieces with 5 cuts, as shown below. Make each cut a diameter, so you only need 5 cuts, not 10.



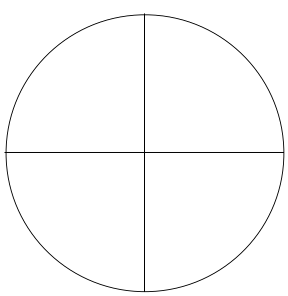
Five cuts is, in fact, the fewest you can use. A trick in later problems of cutting parallel to the table doesn't buy you anything here, since it takes five radial cuts to divide it into five same-size pieces, plus one parallel to the table, to make 10.

Problem 7 *(Student page 125)* To get three same-size pieces, cut the cake into three squares. To get four same-size pieces, make each one L-shaped.





To get 12 same-size pieces, start with the picture for three pieces; cut each one into four smaller squares.

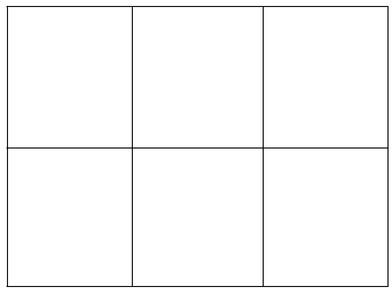


Problem 8 *(Student page 126)* To cut the circular cake into eight same-size pieces with only three cuts, draw two perpendicular diameters, dividing the surface of the cake into fourths. You will cut along these two lines, but you will also make a third cut through the middle of the cake, parallel to the table. This divides the cake into two shorter cakes, each of which is divided into four same-size pieces.

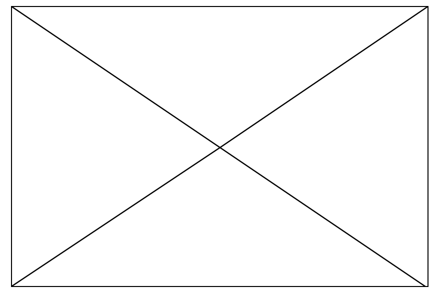
Problem 9 *(Student page 126)* Use three cuts to divide the surface of the rectangular cake into sixths, as shown on the next page. Then make a fourth cut by cutting through the cake, parallel to the table, as in Problem 8. This will produce twelve same-size pieces.

Cut a Rug and Other Dissections (continued)

Alternatively, cut the cake into four pieces with two cuts, and then cut twice (into thirds) parallel to the table. You then have three thin cakes, each divided into four pieces.



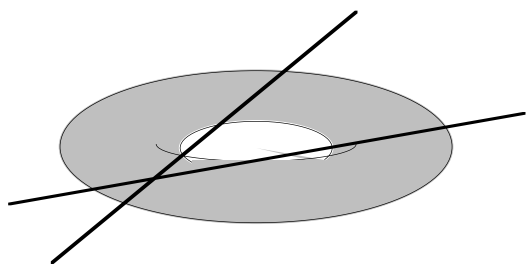
Cut once parallel to the table.



Cut twice parallel to the table.

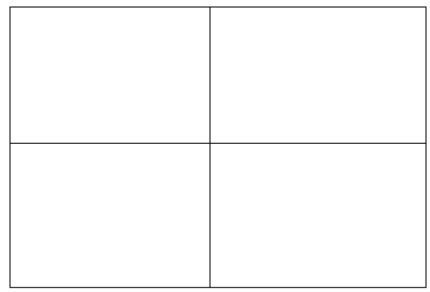
If you can move the pieces between cuts, you can get 18 pieces with only three cuts. Can you figure out how?

Problem 10 *(Student page 126)* The two cuts shown below divide the bagel into five pieces. Now use the third cut to slice the bagel in half, cutting parallel to the table. This will yield ten pieces.

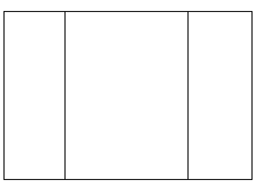


CUT A RUG AND OTHER DISSECTIONS (continued)

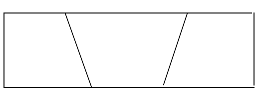
Problem 11 *(Student page 126)* Some of these are easier to adapt than others. For Problem 5, you can create four identical pieces, both in cake and frosting, by connecting opposite midpoints.



Another method involves making cuts that look like this:

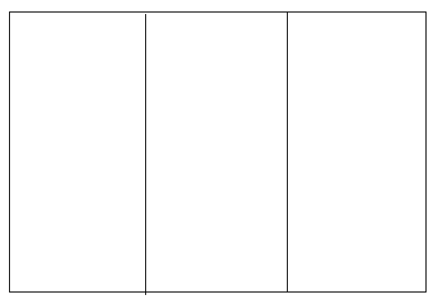


Top view



Side view

Three pieces is more difficult. One method involves creating nine different pieces and dividing them up so that everyone gets three pieces that add up to the same total amount of cake and frosting. Cut into three congruent pieces, parallel to one side, and cut into thirds parallel to the table.

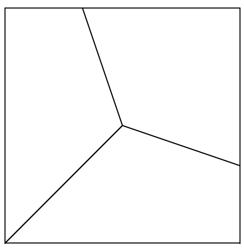


This creates nine pieces, all with the same amount of cake, of four "frosting types": Two *A* pieces have frosting on top, along a long side, and along two short sides. One *B* piece has frosting on top and along two short sides. Two *C* pieces have frosting only along two short sides. And four *D* pieces have frosting along one long side and two short sides. Now divide up the pieces like this:

Two people get one *A*, one *C*, and one *D*. The third person gets one *B* and two *D* pieces. The totals for all three people are the same: They have one top of frosting, six short sides, and two long sides.

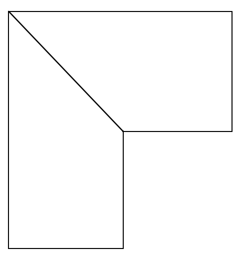
This method isn't very general. We don't know if it will work for five pieces, for example, and it could get messy. It turns out that, if the top of the cake is square instead of rectangular, there is a nice general method to divide the cake into any n pieces with the same amount of cake and frosting: Divide the perimeter into n equal lengths, and then connect the division points to the center of the square.

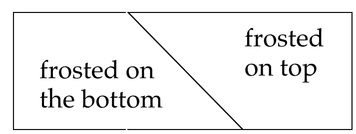
The method for squares for $n = 3$:



For Problem 6, the solution shown already divides both cake and frosting fairly. Circular cakes are particularly nice that way.

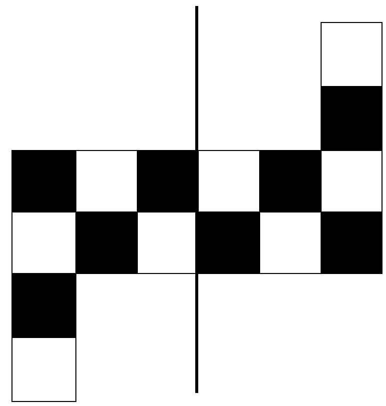
For Problem 7, we can turn it into a rectangle by slicing along the diagonal shown and flipping one piece over.





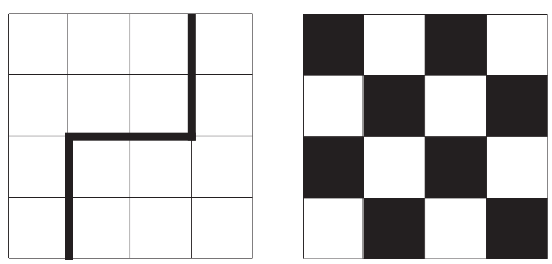
This means half of the cake is frosted on top and the sides, and half is frosted on the bottom and the sides, but none of the frosting overlaps both top and bottom, so the solutions for three and four pieces for the rectangular cake will work here as well. For twelve pieces, we can divide it first into three pieces using the method for rectangular cakes; then divide each of those into four pieces, creating twelve fair pieces.

Problem 12 *(Student page 127)* Cut the piece in two along the line shown below. This divides the figure into two pieces which are the same size and shape.

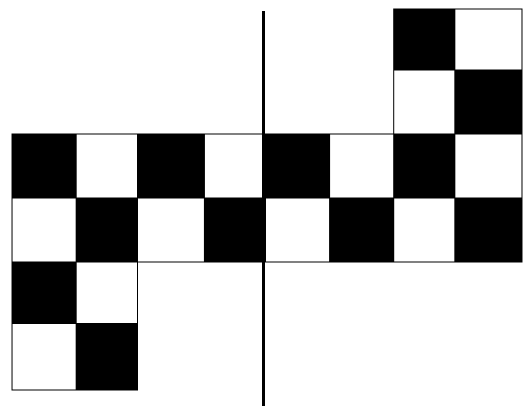


Arrange the two pieces into a 4×4 checkerboard. The picture on the left shows the arrangement of the two pieces, and the picture on the right shows the finished product, with the shading.

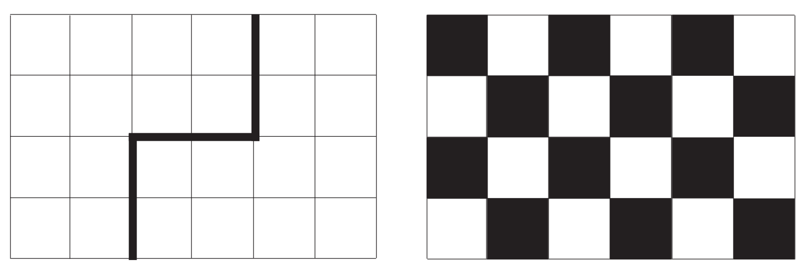
Notice that the black and white squares still alternate!



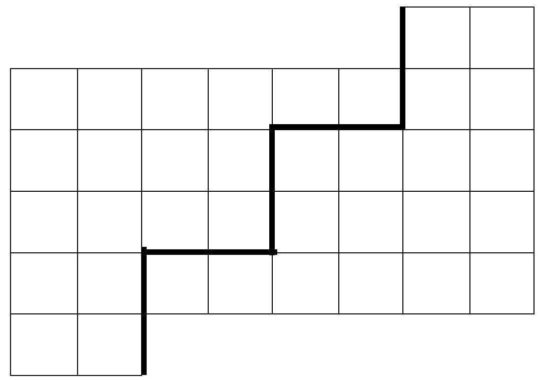
Problem 13 *(Student page 127)* The line below shows how to cut the figure into two pieces.



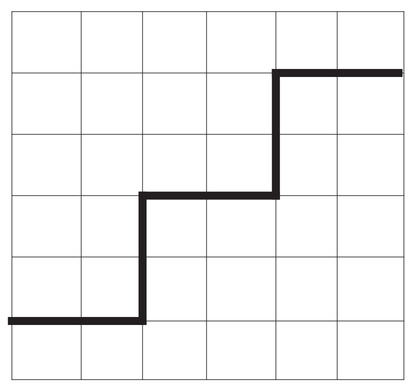
Here's how to arrange them into a 4 × 6 checkerboard:

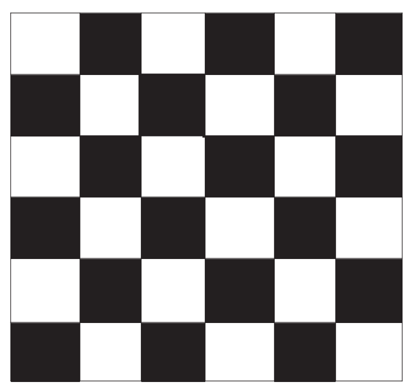


Problem 14 *(Student page 128)* This problem is trickier: The picture below shows how you cut through the original figure. The shading is gone, to make it easier to see how to cut.

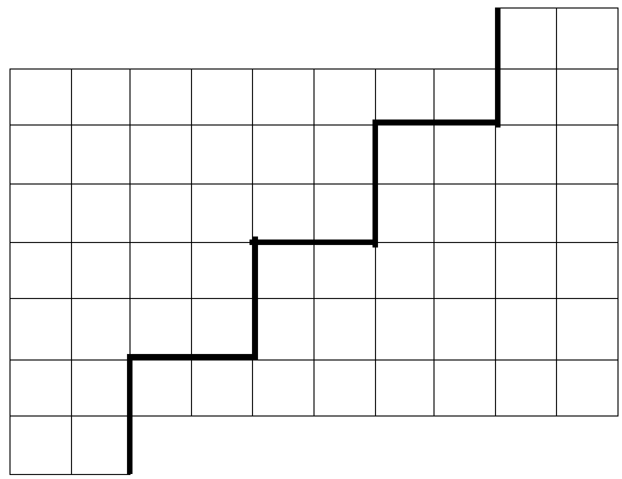


Now take the piece on the right and slide it down two rows and to the left two rows. This will form a 6 × 6 checkerboard.

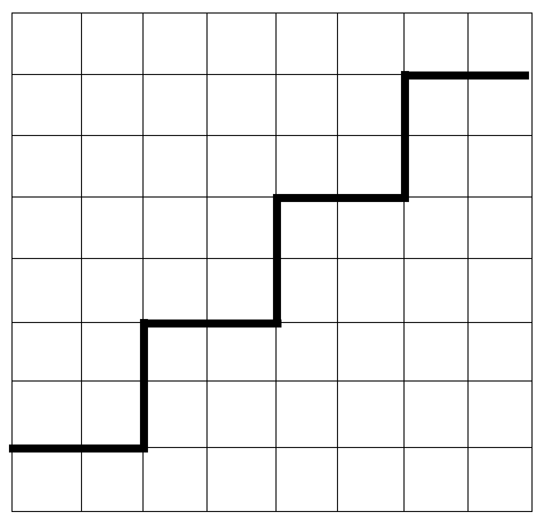


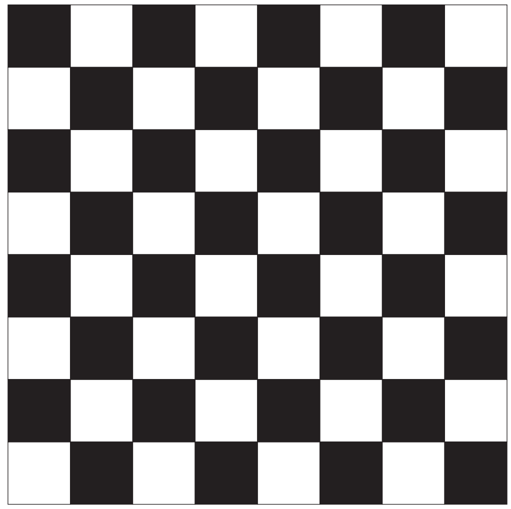


Problem 15 *(Student page 128)* Once again you use zigzag cuts to divide the figure into two shapes which have the same size and shape (the shading is omitted so that you can see the cuts clearly).

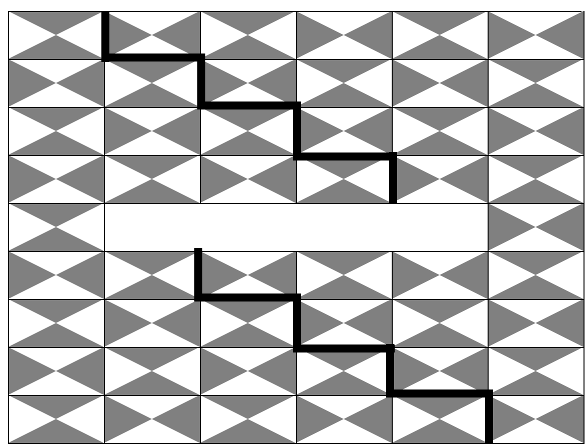


Here's how to arrange the two pieces into an 8×8 checkerboard.

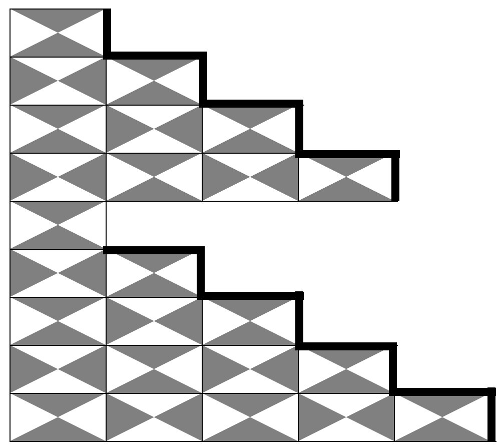


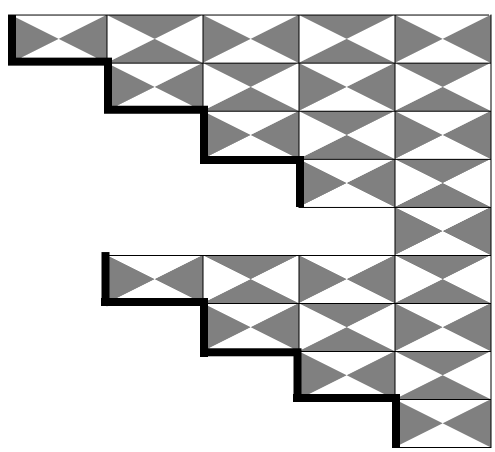


Problem 16 *(Student page 129)* Cut the rug as shown below.



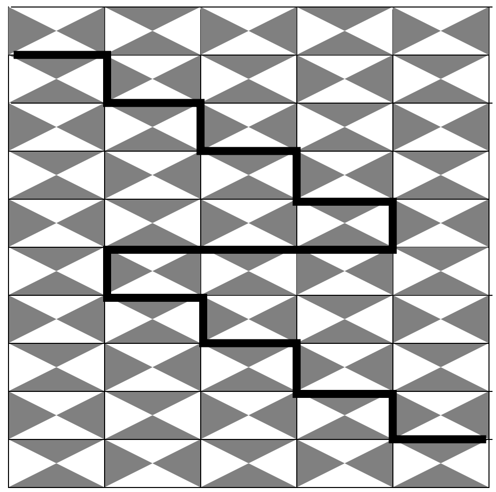
This produces two pieces, which look like the pictures here:



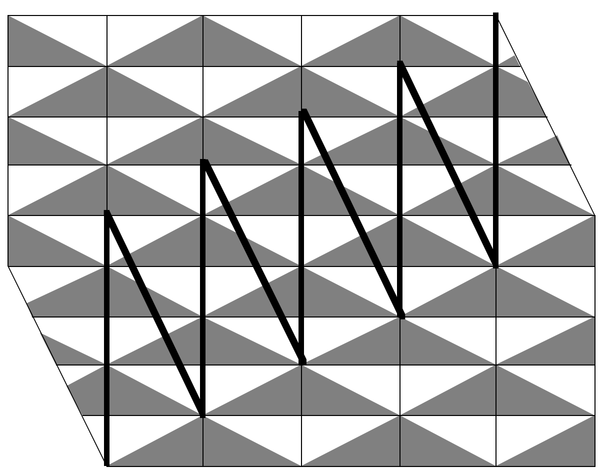


CUT A RUG AND OTHER DISSECTIONS *(continued)*

You can join these two pieces together to form a square. By taking the right-hand figure from the original rug and move it up and to the left. The pieces will fit together like this:

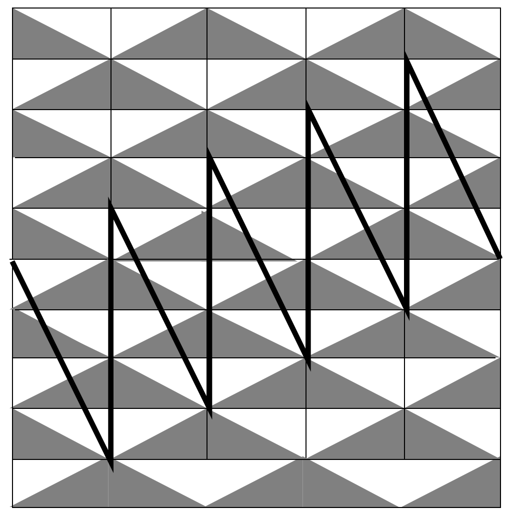


Problem 17 *(Student page 129)* Make a series of spike-shaped cuts, like this:



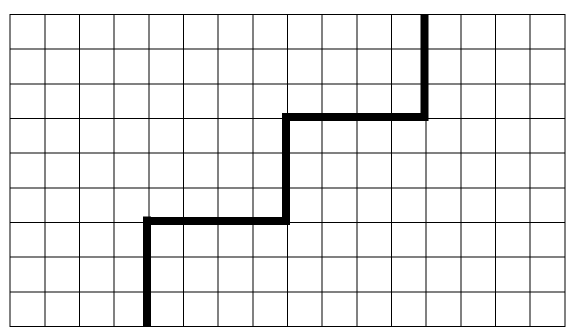
CUT A RUG AND OTHER DISSECTIONS *(continued)*

You can pull apart the two pieces, then slide the bottom piece to the left and down so that each jagged edge fits into the jagged opening to the left of where it came from. This will form a square.



Problem 18 *(Student page 129)*

Yes, it's possible. We want to cut four "squares" from the bottom, and add three "squares" to the side. Cut along the lines shown (this can be done as a single cut).



Then rearrange the pieces like this: